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State-space models 3 from a generic ODE

J A Rossiter

Introduction

The previous videos introduced state space models beginning from 1st principles modelling.

In some cases, a user may be supplied just with an ODE, perhaps high order, but nevertheless it is convenient to express this as an equivalent state space model.

This video introduces a standard process for doing this.

State space modelling

1. The first step in forming a state space model is to define the states. The model is made up of the equations which define the state derivatives.
2. There should be enough independent states to capture the entire system dynamics.
3. In general the number of states required matches the system order, that is how many dynamic modes are in the system behaviour.
4. With an ODE, the order is already known so it remains to identify suitable states.

Consider a generic 2nd order ODE

The main technique is to treat every derivative in the ODE as 1st order derivative, that is the first derivative of some state, to be defined.

The definition of the states is then automatic.

$$ku = a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx$$

$$x_1 = \frac{dx}{dt}$$

$$\frac{dx_1}{dt} = \frac{d^2x}{dt^2}$$

This is the derivative of dx/dt. Therefore define dx/dt as a state, say x_1.

Already a first derivative of x.

State space model for a generic 2nd order ODE

We now have two equations containing first derivatives and two states.

$$ku = a\frac{dx_1}{dt} + bx_1 + cx$$

$$x_1 = \frac{dx}{dt}$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{-b}{a} & \frac{-c}{a} \\ 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x \end{bmatrix}}_{z} + \underbrace{\begin{bmatrix} \frac{k}{a} \\ 0 \end{bmatrix}}_{B} u$$

Consider a generic nth order ODE

The main technique is to treat every derivative in the ODE as 1st order derivative, that is the first derivative of some state, to be defined.

The definition of the states is then automatic.

$$ku = a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_2 \frac{d^2 x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x$$

This is the derivative of $d^{n-1}x/dt^{n-1}$. Let $d^{n-1}x/dt^{n-1} = x_{n-1}$.

This is the derivative of dx/dt. Let $dx/dt = x_1$.

Already a first derivative of x.

Consider a generic nth order ODE

Substitute in the newly defined states.

$$x_{n-1} = \frac{d^{n-1}x}{dt^{n-1}} = \frac{dx_{n-2}}{dt}; \quad x_2 = \frac{d^2x}{dt^2} = \frac{dx_1}{dt}; \quad x_1 = \frac{dx}{dt}$$

$$ku = a_n \frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_2 \frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x$$

$$ku = a_n \frac{dx_{n-1}}{dt} + a_{n-1}x_{n-1} + \cdots + a_2 x_2 + a_1 x_1 + a_0 x$$

Now we have equations defining n 1st order derivatives and n states.

State space model

Use the n equations to form a state space model with n states.

$$x_{n-1} = \frac{dx_{n-2}}{dt}; \quad x_2 = \frac{dx_1}{dt}; \quad x_1 = \frac{dx}{dt}$$

$$ku = a_n \frac{dx_{n-1}}{dt} + a_{n-1}x_{n-1} + \cdots + a_2 x_2 + a_1 x_1 + a_0$$

$$\frac{d}{dt}\begin{bmatrix} x_{n-1} \\ x_{n-2} \\ \vdots \\ x \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{a_{n-1}}{a_n} & -\frac{a_{n-2}}{a_n} & \cdots & -\frac{a_0}{a_n} \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_{n-1} \\ x_{n-2} \\ \vdots \\ x \end{bmatrix}}_{z} + \underbrace{\begin{bmatrix} \frac{k}{a_n} \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{B} u$$

Ordering of state vector

One can choose to stack the vectors in whichever order is most convenient.

The following slides show two alternative state space models for the same system – both are correct.

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4th order state space model

Use the n equations to form a state space model with n states.

$$x_3 = \frac{dx_2}{dt}; \quad x_2 = \frac{dx_1}{dt}; \quad x_1 = \frac{dx}{dt}$$

$$ku = a_4 \frac{dx_3}{dt} + a_3 x_3 + a_2 x_2 + a_1 x_1 + a_0 x$$

$$\frac{d}{dt}\begin{bmatrix} x_3 \\ x_2 \\ x_1 \\ x \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{a_3}{a_4} & -\frac{a_2}{a_4} & -\frac{a_1}{a_4} & -\frac{a_0}{a_4} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_3 \\ x_2 \\ x_1 \\ x \end{bmatrix}}_{z} + \underbrace{\begin{bmatrix} \frac{k}{a_4} \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{B} u$$

4^{th} order state space model

Use the n equations to form a state space model with n states.

$$x_3 = \frac{dx_2}{dt}; \quad x_2 = \frac{dx_1}{dt}; \quad x_1 = \frac{dx}{dt}$$

$$ku = a_4 \frac{dx_3}{dt} + a_3 x_3 + a_2 x_2 + a_1 x_1 + a_0 x$$

$$\frac{d}{dt}\begin{bmatrix} x \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{a_0}{a_4} & -\frac{a_1}{a_4} & -\frac{a_2}{a_4} & -\frac{a_3}{a_4} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{z} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{k}{a_4} \end{bmatrix}}_{B} u$$

Summary

Illustrated state-space model derivation for a generic nth order ODE.

It is noted that an easy selection of states is one whereby each derivative in the ODE is treated as an equivalent 1st order derivative of a state to be defined.

The compact form has vectors of states/inputs and matrices of coefficients.

Different ordering of states leads to different A,B matrices.

$$\dot{x} = Ax + Bu$$

Anthony Rossiter
Department of Automatic Control and Systems Engineering
University of Sheffield
www.shef.ac.uk/acse