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State-space models 4 finding the output

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Introduction

- The previous videos introduced state space models beginning from 1st principles modelling.
- However, so far the focus has been on replicating the dynamics, but little thought has been given to how key values might be extracted.
- This video introduces the concept of the output matrix.

Consider a generic nth order ODE

A generic nth order ODE has an equivalent state space model.

$$ku = a_n \frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_2 \frac{d^2 x}{dt^2} + a_1 \frac{dx}{dt}$$

$$\frac{d}{dt}\begin{bmatrix} x_{n-1} \\ x_{n-2} \\ \vdots \\ x \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{a_{n-1}}{a_n} & -\frac{a_{n-2}}{a_n} & \cdots & -\frac{a_0}{a_n} \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_{n-1} \\ x_{n-2} \\ \vdots \\ x \end{bmatrix}}_{z} + \underbrace{\begin{bmatrix} \frac{k}{a_n} \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{B} u$$

How do I find the values of interest?

Values of interest

In this case, an obvious value of interest is the state x. This is the last value in the 'state vector' z.

Hence:

$$x = \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} x_{n-1} \\ x_{n-2} \\ \vdots \\ x \end{bmatrix}}_{z} = Cz$$

If we were interested in the first derivative, this can equally be determined.

$$\frac{dx}{dt} = x_1 = \begin{bmatrix} 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} x_{n-1} \\ \vdots \\ x_1 \\ x \end{bmatrix} = Cz$$

Values of interest - outputs

If we were interested in the state x and the derivative, we can give the matrix C two rows and define an output vector.

$$y = \begin{bmatrix} x \\ x_1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} x_{n-1} \\ \vdots \\ x_1 \\ x \end{bmatrix}}_{z} = Cz$$

Outputs are defined as values of interest (often but not necessarily specific states) and in general the definition of C follows accordingly.

Generic form

A more complete state space model takes the following form.

$$\dot{x} = Ax + Bu; \qquad y = Cx + Du$$

States

Inputs

Outputs

Although included here for completeness, in many cases the matrix D=0 because otherwise this implies a system whos output can change instantaneously.

Find the state space model of a mass-spring damper with outputs of displacement and velocity

From resource 2 one can find a state space model for the mass spring damper as follows.

$$\begin{bmatrix} \frac{dv}{dt} \\ \frac{dx}{dt} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{-\hat{B}}{M} & \frac{-k}{M} \\ 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} v \\ x \end{bmatrix}}_{z} + \underbrace{\begin{bmatrix} \frac{1}{M} \\ 0 \end{bmatrix}}_{B} f$$

The output is:

$$\begin{bmatrix} v \\ x \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} v \\ x \end{bmatrix}}_{z} = Cz$$

Find the state space model of a DC servo with outputs of displacement and current.

From resource 2 one can find a state space model for the mass spring damper as follows.

$$\underbrace{\begin{bmatrix} \frac{dw_L}{dt} \\ \frac{di}{dt} \\ \frac{d\theta_L}{dt} \end{bmatrix}}_{} = \underbrace{\begin{bmatrix} -\frac{\hat{B}}{J} & \frac{k}{J} & 0 \\ -\frac{k}{L} & -\frac{R}{L} & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} w_L \\ i \\ \theta_L \end{bmatrix}}_{z} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{L} \\ 0 \end{bmatrix}}_{B} v$$

The output is:

$$\begin{bmatrix} \theta_L \\ i \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} w_L \\ i \\ \theta_L \end{bmatrix}$$

Summary

A more general state space model includes two vector equations.

1. One vector equation defines the system dynamics through first order derivatives (using A,B matrices).
2. The second vector equation defines the outputs y through matrices C,D. This equation has no dynamics.

$$\dot{x} = Ax + Bu; \quad y = Cx + Du$$

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