The University Of Sheffield.

State-space models 5
transfer functions from state space

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Introduction

The previous videos introduced state space models.

It is useful to consider to what extent these can be represented by an equivalent transfer function model.

This video introduces a standard process for doing this.

Background

A general state space model includes two vector equations.

1. One vector equation defines the system dynamics through first order derivatives (using A,B matrices).
2. The second vector equation defines the outputs y through matrices C,D. This equation has no dynamics.

$$\dot{x} = Ax + Bu; \qquad y = Cx + Du$$

Laplace Transforms

It is expected that viewers know the basics of Laplace and therefore recognise expressions such as:

$$L[x(t)] = X(s) \quad \Rightarrow \quad L[\frac{dx}{dt}] = sX(s) - x(0)$$

Here we are dealing with transfer function models and so initial conditions will be ignored.

Apply this definition to the state space model.

Laplace transforms of model

Take Laplace of every term. Use capitals for Laplace and lower case for time domain.

$$\dot{x} = Ax + Bu; \quad y = Cx + Du$$

$$sX = AX + BU; \quad Y = CX + DU$$

Rearrange to solve for X(s) and Y(s)

$$(sI - A)X = BU \quad \Rightarrow \quad \begin{aligned} X &= (sI - A)^{-1}BU \\ Y &= [C(sI - A)^{-1}B + D]U \end{aligned}$$

Example 1

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x \end{bmatrix} = \underbrace{\begin{bmatrix} -b & -c \\ 1 & 0 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x_1 \\ x \end{bmatrix}}_{z} + \underbrace{\begin{bmatrix} k \\ 0 \end{bmatrix}}_{B} f$$

$$G = (sI - A)^{-1}B = \frac{\begin{bmatrix} s & -c \\ 1 & s+b \end{bmatrix}\begin{bmatrix} k \\ 0 \end{bmatrix}}{(s+b)s+c} = \frac{\begin{bmatrix} ks \\ k \end{bmatrix}}{s^2+bs+c}$$

Example 2 (mass-spring-damper)

$$\begin{bmatrix} \frac{dv}{dt} \\ \frac{dx}{dt} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{-\hat{B}}{M} & \frac{-k}{M} \\ 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} v \\ x \end{bmatrix}}_{z} + \underbrace{\begin{bmatrix} \frac{1}{M} \\ 0 \end{bmatrix}}_{B} f$$

$$\begin{bmatrix} v \\ x \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} v \\ x \end{bmatrix}}_{z} = Cz$$

$$G = C(sI - A)^{-1}B = \frac{\begin{bmatrix} s & -\frac{k}{M} \\ 1 & s+\frac{\hat{B}}{M} \end{bmatrix} \begin{bmatrix} \frac{1}{M} \\ 0 \end{bmatrix}}{(s+\frac{\hat{B}}{M})s + \frac{k}{M}} = \frac{\begin{bmatrix} \frac{s}{M} \\ \frac{1}{M} \end{bmatrix}}{s^2 + \frac{\hat{B}}{M}s + \frac{k}{M}}$$

REMARK

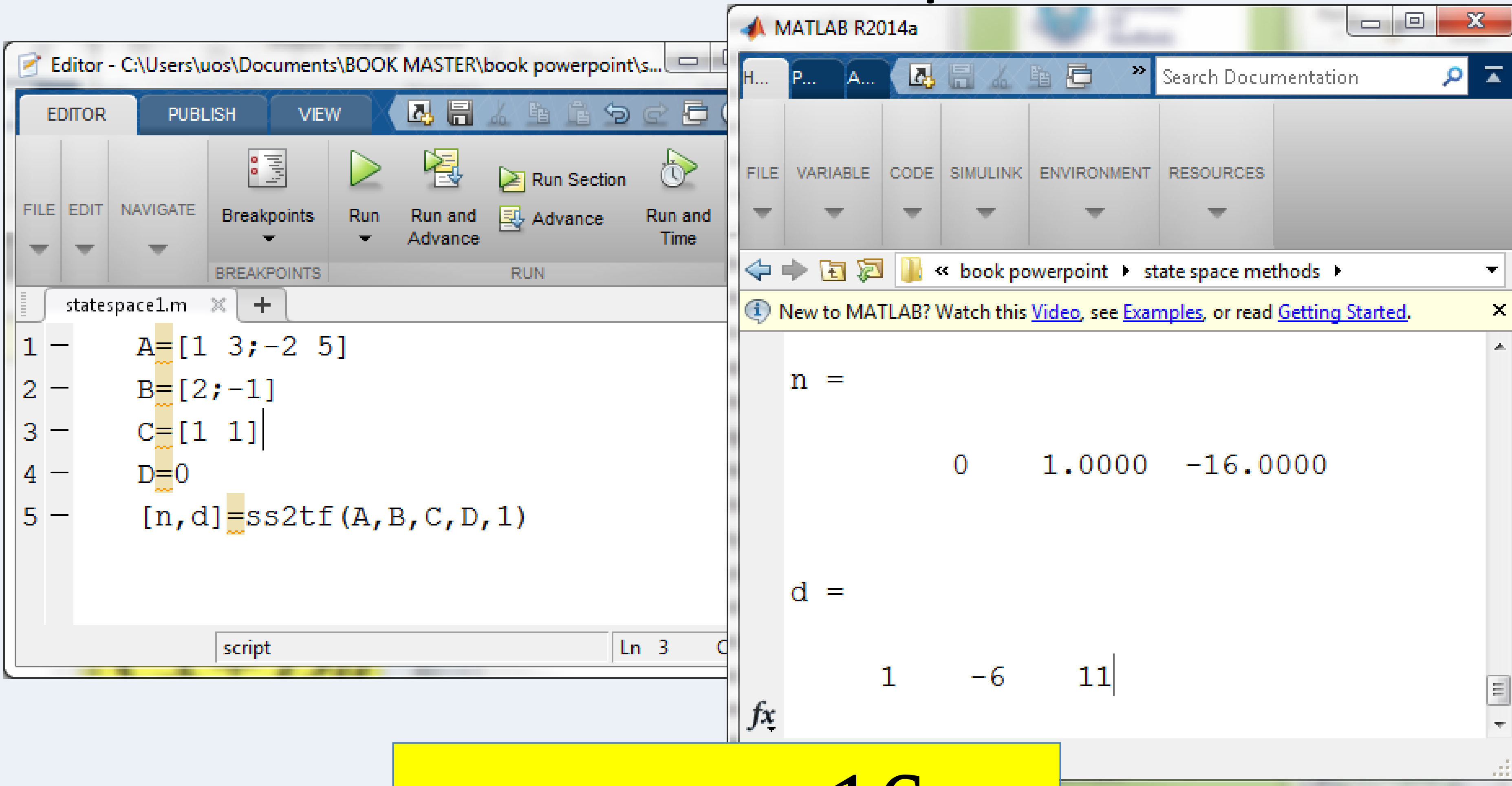
Finding transfer functions by hand is not recommended in general as it is rather tedious.

Instead, it is recommended that students use a computer.

Some examples with MATLAB are given next.

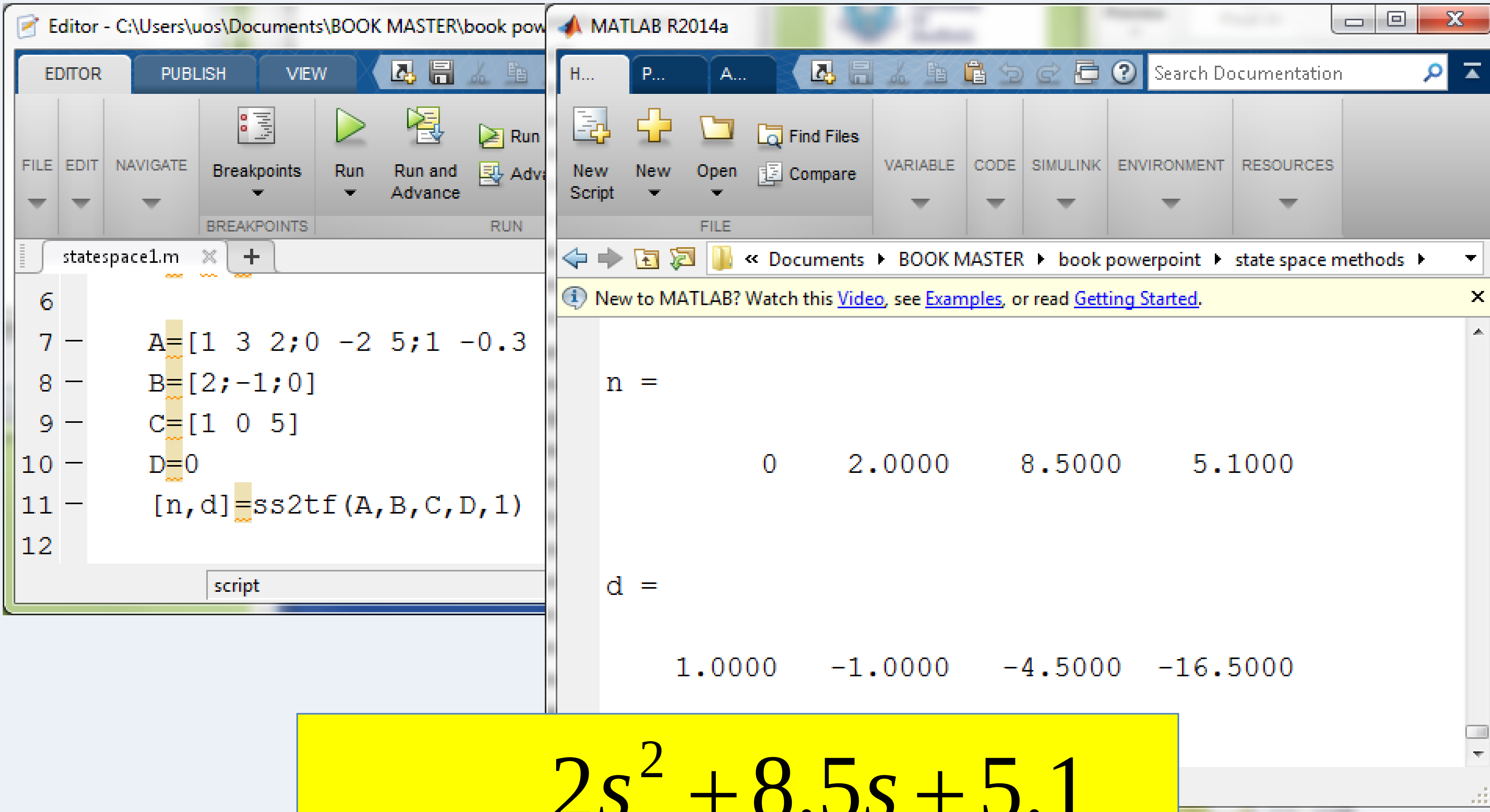
NOTE: MATLAB will only do one input at a time

MATLAB example 1



$$G = \frac{s-16}{s^2-6s+11}$$

MATLAB example 2



$$G=\frac{2s^2+8.5s+5.1}{s^3-s^2-4.5s-16.5}$$

REMARK 2

The denominator of the transfer function has a clear analogy to the eigenvalues of A, hence it is clear that the eigenvalues of A must be the system poles.

The denominator of a matrix inverse is the determinant.

$$Y = [C(sI - A)^{-1}B + D]U \quad \Rightarrow \quad poles \equiv |sI - A| = 0$$

$$eigenvalues \equiv |\lambda I - A| = 0$$

Summary

Illustrated that it is straightforward in principle to find a transfer function model from a state-space model.

This is not a pen and paper exercise in general due to the requirement of a matrix inverse – software tools are recommended.

$$\dot{x} = Ax + Bu; \qquad y = Cx + Du$$

$$(sI - A)X = BU \quad \Rightarrow \quad \begin{array}{c} X = (sI - A)^{-1}BU \\ Y = [C(sI - A)^{-1}B + D]U \end{array}$$

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