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State-space models 6
conversion from a transfer function

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Introduction

1. The previous video showed how to form a transfer function model from a state space model.
2. This video looks at the reverse process, that is finding a state space model from a transfer function model.
3. Viewers are reminded that a state space representation is not unique. Several alternative state space forms can represent the same transfer function model.
4. Canonical forms can be particular useful.

Assumption

For ease of algebra denominator polynomials will be monic.

It is always possible to achieve this by scaling all coefficients as required.

MONIC means the coefficient of the maximum power is 1.

Basic concept

The approach here will begin by deriving the results for a transfer function with a constant numerator.

$$X(s) = \frac{k}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} U(s)$$

This clearly analogous to an ODE – see video 3!

$$ku = \frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_2 \frac{d^2 x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x$$

The next 2 slides summarise the results of video 3

State space model

Use the n equations to form a state space model with n states.

$$x_{n-1} = \frac{dx_{n-2}}{dt}; \quad x_2 = \frac{dx_1}{dt}; \quad x_1 = \frac{dx}{dt}$$

$$ku = \frac{dx_{n-1}}{dt} + a_{n-1}x_{n-1} + \cdots + a_2 x_2 + a_1 x_1 + a_0 x$$

$$\frac{d}{dt}\begin{bmatrix} x_{n-1} \\ x_{n-2} \\ \vdots \\ x \end{bmatrix} = \underbrace{\begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_{n-1} \\ x_{n-2} \\ \vdots \\ x \end{bmatrix}}_{z} + \underbrace{\begin{bmatrix} k \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{B} u$$

4^{th} order state space model

Use the n equations to form a state space model with n states.

$$x_3 = \frac{dx_2}{dt}; \quad x_2 = \frac{dx_1}{dt}; \quad x_1 = \frac{dx}{dt}$$

$$ku = \frac{dx_3}{dt} + a_3 x_3 + a_2 x_2 + a_1 x_1 + a_0 x$$

$$\frac{d}{dt}\begin{bmatrix} x \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{z} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ k \end{bmatrix}}_{B} u$$

The ordering of states is a user choice and this changes the implied A, B matrices.

REMARKS

The two examples just given, which simply reverse the order the states are listed, are denoted as

CONTROLLABLE CANONICAL FORMS

A KEY POINT is that the parameters of the ODE (equivalently transfer function denominator) appear explicitly in the A matrix and thus the conversion is straightforward.

Proposal

Given we can find a state space model for a generic ODE.

$$ku = \frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_2\frac{d^2 x}{dt^2} + a_1\frac{dx}{dt} + a_0 x$$

A possible state space model for the equivalent transfer function must be same!

$$X(s) = \frac{k}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}U(s)$$

Find a control canonical form for the following

$$X(s) = \frac{4}{s^3 + 2s^2 + s + 3} U(s)$$

$$\frac{d}{dt}[z] = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -1 & -2 \end{bmatrix}}_{A} z + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}}_{B} u; \qquad z = \begin{bmatrix} x \\ \dot{x} \\ \ddot{x} \end{bmatrix}$$

$$x = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{C} z$$

EXTENSIONS TO MORE COMPLICATED NUMERATORS

REMARKS

The state space model derived corresponds to:

$$X(s) = \frac{1}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} U(s)$$

Imagine a similar transfer function with an output which is the derivative of X(s):

$$W(s) = \frac{s}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} U(s) \quad \Rightarrow \quad w = \frac{dx}{dt}$$

Then note that the state 'w', was already defined in the state space model for X(s) so can be extracted as an output.

Find a transfer function model which has as an output all the states.

$$\frac{d}{dt}\begin{bmatrix} x \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{z} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{B} u$$

$$X = (sI - A)^{-1} BU$$

$$x_n = \frac{d^n x}{dt^n}$$

$$x_2 = \frac{d^2 x}{dt^2}$$

$$x_1 = \frac{dx}{dt}$$

$$\begin{bmatrix} X(s) \\ X_1(s) \\ \vdots \\ X_{n-1}(s) \end{bmatrix} = \begin{bmatrix} \dfrac{1}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \\ \dfrac{s}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \\ \vdots \\ \dfrac{s^{n-1}}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \end{bmatrix} U(s)$$

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Find a state space model for the following

It should be clear that this amounts to only a change in the C matrix, that is in which the 2nd state is exported as the output.

$$W(s) = \frac{s}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} U(s); \quad w = \frac{dx}{dt}$$

$$W(s) = \begin{bmatrix} 0 & 1 & 0 & \cdots \end{bmatrix} \begin{bmatrix} X(s) \\ X_1(s) \\ \vdots \\ X_{n-1}(s) \end{bmatrix} = \frac{s}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} U(s)$$

Find a state space model for

It should be clear that this amounts to only a change in the C matrix, that is in which the rth state is exported as the output.

$$W(s) = \frac{s^r}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} U(s); \quad w = \frac{d^r x}{dt^r}$$

$$W(s) = \begin{bmatrix} 0 & 0 & \cdots & 1 & \cdots & 0 \end{bmatrix} \begin{bmatrix} X(s) \\ X_1(s) \\ \vdots \\ X_r(s) \\ \vdots \\ X_{n-1}(s) \end{bmatrix} = \frac{s^r}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} U(s)$$

Find a control canonical form for the following

$$X(s) = \frac{4s}{s^3 + 2s^2 + s + 3} U(s)$$

NOTE: Placed 4 in the C matrix instead of B.

$$\frac{d}{dt}[z] = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -1 & -2 \end{bmatrix}}_{A} z + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B} u; \qquad z = \begin{bmatrix} x \\ \dot{x} \\ \ddot{x} \end{bmatrix}$$

$$x = \underbrace{\begin{bmatrix} 0 & 4 & 0 \end{bmatrix}}_{C} z$$

NOTE: Numerator coefficient is now in the C matrix!

Find a control canonical form for the following

$$Y(s)=\frac{6s^2+4s+2}{s^4-s^3+2s^2+s+3}U(s)$$

$$\frac{d}{dt}[z]=\underbrace{\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\-3&-1&-2&1\end{bmatrix}}_{A}z+\underbrace{\begin{bmatrix}0\\0\\1\end{bmatrix}}_{B}u;\qquad y=\underbrace{\begin{bmatrix}2&4&6&0\end{bmatrix}}_{C}z$$

NOTE: The only change is to the C matrix which has all numerator coefficients!

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Find a control canonical form for the following

$$Y(s)=\frac{b_{n-1}s^{n-1}+\cdots+b_1 s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_2 s^2+a_1 s+a_0}U(s)$$

$$\frac{d}{dt}[z]=\underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}}_{A} z+\underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}}_{B} u;$$

$$y=\underbrace{\begin{bmatrix} b_0 & b_1 & \cdots & b_{n-1} \end{bmatrix}}_{C} z$$

NOTE: States can be in reverse order with obvious impact on matrix definitions!

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Find a control canonical form for the following

$$Y(s)=\frac{b_{n-1}s^{n-1}+\cdots+b_1 s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_2 s^2+a_1 s+a_0}U(s)$$

$$\frac{d}{dt}[z]=\underbrace{\begin{bmatrix}-a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0\\ 1 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & 0\end{bmatrix}}_{A}z+\underbrace{\begin{bmatrix}1\\0\\ \vdots\\0\end{bmatrix}}_{B}u;$$

$$y=\underbrace{\begin{bmatrix}b_{n-1} & b_{n-2} & \cdots & b_0\end{bmatrix}}_{C}z$$

Summary

Illustrated how state space models can be derived from transfer functions.

Emphasised the control canonical form as these can be constructed by inspection.

Viewers should remember however that state space models are not unique and different choices of states will lead to different A,B,C matrices.

We have not covered the observer canonical form which will come up later when we discuss observers.

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