State-space models 7 diagonal canonical form

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Introduction

1. The previous video showed how to form a a state space model from a transfer function using the control canonical form.
2. Where a system has only real and distinct poles, one alternative is the diagonal canonical form.
3. Being diagonal, this has some advantages although the states have less meaning and partial fractions are required.

Partial fractions

Assuming real and distinct poles, a partial fraction expansion takes the following form.

$$G = \frac{b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_2 s^2 + a_1 s + a_0}$$

$$G = \frac{b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

$$G = \frac{R_1}{(s+p_1)} + \frac{R_2}{(s+p_2)} + \cdots + \frac{R_n}{(s+p_n)}$$

State-space from partial fraction

Define a separate, independent state for each partial fraction term.

$$G = \frac{R_1}{(s+p_1)} + \frac{R_2}{(s+p_2)} + \cdots + \frac{R_n}{(s+p_n)}$$

$$\frac{dx_1}{dt} = -p_1 x_1 + u; \quad \frac{dx_2}{dt} = -p_2 x_2 + u; \quad \cdots$$

NOTE: Ignore the residues for now and reintroduce in the output equation!

$$y = R_1 x_1 + R_2 x_2 + \cdots$$

Example state space model

By inspection.

$$G=\frac{R_1}{(s+p_1)}+\frac{R_2}{(s+p_2)}+\cdots+\frac{R_n}{(s+p_n)}$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \underbrace{\begin{bmatrix} -p_1 & 0 & \cdots & 0 \\ 0 & -p_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -p_n \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{z} + \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}_{B} u$$

$$y=\underbrace{\begin{bmatrix} R_1 & R_2 & \cdots & R_n \end{bmatrix}}_{C} z$$

Example

Find a state space model for the following.

First do a partial fraction expansion.

$$G=\frac{3s+1}{s^3+6s^2+11s+6}=\frac{3s+1}{(s+1)(s+2)(s+3)}$$

$$G=\frac{-1}{(s+1)}+\frac{5}{(s+2)}+\frac{-4}{(s+3)}$$

$$\frac{d}{dt}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\underbrace{\begin{bmatrix}-1&0&0\\0&-2&0\\0&0&-3\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}}_{z}+\underbrace{\begin{bmatrix}1\\1\\1\end{bmatrix}}_{B}u;\qquad y=\underbrace{\begin{bmatrix}-1&5&-4\end{bmatrix}}_{C}z$$

REPEATED ROOTS AND JORDAN FORMS

Repeated roots

Where a partial fraction has repeated roots a slightly modified, so called Jordan form, is required.

It is easier to illustrate this first with some simple examples.

$$G = \frac{R_1}{(s+a)^2} + \frac{R_2}{s+a};$$

$$H = \frac{R_1}{(s+a)^3} + \frac{R_2}{(s+a)^2} + \frac{R_3}{s+a};$$

State space form for repeated poles

A suitable A and B matrix to create the required forms is simply stated here.

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -a & 1 \\ 0 & -a \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{z} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u; \qquad y = \underbrace{\begin{bmatrix} R_1 & R_2 \end{bmatrix}}_{C} z$$

$$(sI - A)^{-1}B = \begin{bmatrix} s+a & -1 \\ 0 & s+a \end{bmatrix}^{-1} B = \frac{\begin{bmatrix} s+a & 1 \\ 0 & s+a \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}}{(s+a)^2} = \begin{bmatrix} \dfrac{1}{(s+a)^2} \\ \dfrac{1}{(s+a)} \end{bmatrix}$$

$$C(sI - A)^{-1}B = \frac{R_1}{(s+a)^2} + \frac{R_2}{(s+a)}$$

State space form for repeated poles

3rd order is similar to 2nd order.

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -a & 1 & 0 \\ 0 & -a & 1 \\ 0 & 0 & -a \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{z} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B} u; \quad y = \underbrace{\begin{bmatrix} R_1 & R_2 & R_3 \end{bmatrix}}_{C} z$$

$$(sI-A)^{-1}B = \begin{bmatrix} -a & 1 & 0 \\ 0 & -a & 1 \\ 0 & 0 & -a \end{bmatrix}^{-1} B = \frac{\begin{bmatrix} (s+a)^2 & s+a & 1 \\ 0 & (s+a)^2 & s+a \\ 0 & 0 & (s+a)^2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}{(s+a)^3} = \begin{bmatrix} \frac{1}{(s+a)^3} \\ \frac{1}{(s+a)^2} \\ \frac{1}{(s+a)} \end{bmatrix}$$

$$C(sI-A)^{-1}B = \frac{R_1}{(s+a)^3} + \frac{R_2}{(s+a)^2} + \frac{R_3}{(s+a)}$$

Find a diagonal/Jordan canonical form for the following

$$X(s) = \frac{4s+2}{s^3+4s^2+5s+2}U(s)$$

First do the partial fraction expansion.

$$G = \frac{6}{(s+1)^2} + \frac{-2}{(s+1)} + \frac{-6}{(s+2)}$$

$$\frac{d}{dt}[z] = \underbrace{\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}}_{A} z + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}_{B} u; \quad x = \underbrace{\begin{bmatrix} 6 & -2 & -6 \end{bmatrix}}_{C} z$$

Summary

Illustrated how diagonal state space models can be derived from transfer functions with real and distinct roots by first doing a partial fraction expansion.

Demonstrated the Jordon form for real and repeated roots.

Diagonal forms would not be suitable for complex poles.

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