State-space models 8 state transformations

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Introduction

1. The previous videos have demonstrated numerous mechanisms for creating state space models to represent systems.

$$\dot{x} = Ax + Bu; \quad y = Cx$$

2. However, it is clear that a state space model is not unique in that several alternative choices for A,B,C can represent the same system.
3. Implicitly different A,B,C imply different states and often states have no physical interpretation.

Purpose

This video looks at the relationships between state space models which represent the same system.

Also, how do we know if the following represent the same system?

$$\dot{x} = Ax + Bu; \quad y = Cx$$

$$\dot{z} = \hat{A}z + \hat{B}u; \quad y = \hat{C}z$$

State transformation

Consider a simple state transformation T (implicitly T is full rank) and the effect it has on a model.

$$z = Tx; \quad x = T^{-1}z$$

$$\{\dot{x} = Ax + Bu\} \quad \Rightarrow \quad \{T^{-1}\dot{z} = AT^{-1}z + Bu\}$$

$$\Rightarrow \quad \left\{ TT^{-1}\dot{z} = \underbrace{TAT^{-1}}_{\hat{A}}z + \underbrace{TB}_{\hat{B}}u \right\}$$

$$\Rightarrow \quad \{\dot{z} = \hat{A}z + \hat{B}u\}$$

$$y = Cx$$

$$\Rightarrow \quad y = \underbrace{CT^{-1}}_{\hat{C}}z$$

Remarks

- Different choices of T can be used to produce different canonical forms from each other or indeed to specify other state definitions that may be beneficial.
- A common choice of transformation is one which reveals the system modes/poles. This is analogous to the diagonal canonical form.
- Two state space models represent the same Input/ouput system dynamics if there exists a full rank T such that.

$$TAT^{-1} = \hat{A}; \quad TB = \hat{B}; \quad CT^{-1} = \hat{C}$$

EIGENVALUES/VECTORS

Transformation with eigenvectors

One common transformation is the matrix of eigenvectors of the A matrix.

This produces a diagonal form which separates each dynamic mode into independent states.

$$A = W \Lambda W^{-1}$$

$$T = W^{-1}$$

Eigenvectors

Eigenvalues

Eigenvalue/vector decomposition

Use T=W^{-1}.

$$\dot{x} = Ax + Bu;$$

$$y = Cx$$

$$TT^{-1}\dot{z} = \underbrace{TAT^{-1}}_{\hat{A}} z + \underbrace{TB}_{\hat{B}} u$$

$$y = \underbrace{CT^{-1}}_{\hat{C}} z$$

$$\dot{z} = \underbrace{W^{-1}AW}_{\hat{A}} z + \underbrace{W^{-1}B}_{\hat{B}} u; \quad y = \underbrace{CW}_{\hat{C}} z$$

$$\dot{z} = \Lambda z + \hat{B}u; \quad y = \hat{C}z$$

This form can be useful for control design.

INVARIANCE OF BEHAVIOUR UNDER TRANSFORMATION

Behaviour

Given that T is full rank, one can state by inspection that the following must have the same input/output behaviour because otherwise one would end up with an inconsistency in an implied linear equality.

$$\dot{x} = Ax + Bu;$$

$$y = Cx$$

$$TT^{-1}\dot{z} = \underbrace{TAT^{-1}}_{\hat{A}} z + \underbrace{TB}_{\hat{B}} u$$

$$y = \underbrace{CT^{-1}}_{\hat{C}} z$$

This observation is useful later when we determine behaviours more explicitly.

Invariance of eigenvalues

Later it will be shown that the behaviour is linked directly to the eigenvalues of matrix A.

We know from the previous slide that behaviour is invariant and therefore, we expect the eigenvalues to be invariant.

$$\lambda(A) = \lambda(TAT^{-1})$$

It will be demonstrated briefly that this condition holds when T is full rank.

Invariance of eigenvalues (b)

Eigenvalues are defined from a determinant.

$$\lambda(A): \qquad |\lambda I - A| = 0$$

$$\lambda(TAT^{-1}): \quad |\lambda I - TAT^{-1}| = 0$$

Rearrange lower identify using $TT^{-1}=I$.

$$|\lambda I - TAT^{-1}| = 0 \quad \equiv \quad |\lambda TT^{-1} - TAT^{-1}| = 0$$

$$\equiv \quad |T(\lambda I - A)T^{-1}| = 0$$

Remember T is full rank, |T|≠0!

Remark – repeated roots

We will not extend the analysis to Jordan forms as this is more of academic interest than offering new insight.

Suffice to say that equivalent analysis is possible.

Summary

1. Illustrated how a simple state transformation does not change the input/output behaviour.
2. State space matrices, for a given system, are not unique. The user can select the states, and by inference the A,B,C matrices, that are most convenient for the required use.
3. A state transformation does not change the eigenvalues of the state transformation matrix.
4. The states of a transformed system may have no obvious physical interpretation.

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