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State-space feedback 2 pole placement with canonical forms

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Introduction

- The first video introduced the concept of state feedback and demonstrated that it moves the poles.

$$\left\{\begin{matrix} \dot{x} = Ax + Bu \\ u = -Kx \end{matrix}\right\} \quad \Rightarrow \quad \dot{x} = \underbrace{(A - BK)}_{\Phi} x$$

- However, selection of K to achieve specified poles was also shown to be non simple in that a brute force computation of the eigenvalues of (A-BK) was not easy with the parameters of K unknown.
- This video shows how the computation is easy using control canonical forms and moreover, one can place all the poles.

Control canonical form

$$Y(s) = \frac{b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_2 s^2 + a_1 s + a_0} U(s)$$

$$\frac{d}{dt}[z] = \underbrace{\begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}}_{A} z + \underbrace{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{B} u;$$

$$y = \underbrace{\begin{bmatrix} b_{n-1} & b_{n-2} & \cdots & b_0 \end{bmatrix}}_{C} z$$

Observation

In control canonical form, the eigenvalues are determined solely by the parameters along the top row of the A matrix.

$$A=\begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

$$|\lambda I - A| = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0$$

Corollary: One can place the poles precisely if one can select the top row of A precisely without changing any other coefficients.

Impact of state feedback

Consider the impact of state feedback on a control canonical form.

$$B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}; \quad K = \begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix}$$

BK has non-zero terms only in its top row!

$$BK = \begin{bmatrix} k_1 & k_2 & \cdots & k_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

The top row coefficients are independent of each other.

Impact of state feedback on a canonical form.

One can now form the closed-loop state space model by inspection.

$$\left\{\begin{array}{l} \dot{x} = Ax + Bu \\ u = -Kx \end{array}\right\} \quad \Rightarrow \quad \dot{x} = \underbrace{(A - BK)}_{\Phi} x$$

$$A - BK = \begin{bmatrix} -a_{n-1} - k_1 & -a_{n-2} - k_2 & \cdots & -a_1 - k_{n-1} & -a_0 - k_n \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

$$|\lambda I - A + BK| = \lambda^n + (a_{n-1} + k_1)\lambda^{n-1} + \cdots + (a_0 + k_n)$$

One can choose the parameters of the closed-loop pole polynomial directly by choosing the parameters k_i.

EXAMPLES

Example 1: Choose K to set the closed-loop poles at -1 and -2.

$$A=\begin{bmatrix}1 & 2\\ 1 & 0\end{bmatrix};\quad B=\begin{bmatrix}1\\ 0\end{bmatrix};\quad K=\begin{bmatrix}k_1 & k_2\end{bmatrix}$$

$$p_c = s^2+3s+2$$

Form the closed-loop state transition matrix.

$$A-BK=\begin{bmatrix}1-k_1 & 2-k_2\\ 1 & 0\end{bmatrix}$$

$$1-k_1=-3$$
$$2-k_2=-2$$

$$|\lambda I-(A-BK)| = \lambda^2-(1-k_1)\lambda-(2-k_2)=0$$
$$|\lambda I-(A-BK)| = \lambda^2+3\lambda+2$$

Define actual pole polynomial and match coefficients to the desired p_c.

Example 2: Choose K to set the closed-loop poles at -0.5,-1,-1.5.

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \quad K = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$$

$$p_c = s^3 + 3s^2 + 2.75s + 0.75$$

Form the closed-loop state transition matrix.

$$A - BK = \begin{bmatrix} 1-k_1 & 3-k_2 & 3-k_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Define actual pole polynomial and match coefficients to p_c.

$$|\lambda I - (A-BK)| = \lambda^3 - (1-k_1)\lambda^2 - (3-k_2)\lambda - (3-k_3) = 0$$

$$|\lambda I - (A-BK)| = \lambda^3 + 3\lambda^2 + 2.75\lambda + 0.75$$

$$1-k_1 = -3; \quad 3-k_2 = -2.75; \quad 3-k_3 = -0.75$$

Summary $u = -Kx$

Introduced concepts of state feedback with a control canonical form.

1. When a system is in control canonical form, every coefficient of the closed-loop pole polynomial can be defined as desired using state feedback.
2. This means every closed-loop pole can be placed exactly as desired.
3. Viewers should note that this does not imply knowledge of good places to put the poles. In general selecting fast poles may not imply good overall behaviour.

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