State-space feedback 3 transformation to get a canonical form

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Introduction

- The previous video showed how state feedback can place poles precisely when the system is in control canonical form.

$$\left\{\begin{array}{c} \dot{x} = Ax + Bu \\ u = -Kx \end{array}\right\} \quad \Rightarrow \quad \dot{x} = \underbrace{(A - BK)}_{\Phi} x$$

- More generally, the system is not in canonical form, but we may still wish to place the closed-loop poles.
- This video shows how the a transformation can create the canonical form, and thus implicitly allows explicit and straightforward pole placement.

Control canonical form

$$\frac{d}{dt}[z] = \underbrace{\begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}}_{A} z + \underbrace{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{B} u;$$

$$y = \underbrace{\begin{bmatrix} b_{n-1} & b_{n-2} & \cdots & b_0 \end{bmatrix}}_{C} z$$

Remark: A transformation to control canonical form only exists if the system is fully controllable.

Observation

In control canonical form, the eigenvalues are determined solely by the parameters along the top row of the A matrix.

$$A = \begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

$$|\lambda I - A| = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0$$

Corollary: One can place the poles precisely if one can select the top row of A precisely without changing any other coefficients.

Similarity transformation (chapter 1)

Consider how the state feedback varies for two alternative representations linked by a similarity transform.

$$z = Tx; \quad x = T^{-1}z$$

$$\{\dot{x} = Ax + Bu\} \quad \Rightarrow \quad \{T^{-1}\dot{z} = AT^{-1}z + Bu\}$$

$$\Rightarrow \quad \left\{TT^{-1}\dot{z} = \underbrace{TAT^{-1}}_{\hat{A}}z + \underbrace{TB}_{\hat{B}}u\right\}$$

$$\Rightarrow \quad \{\dot{z} = \hat{A}z + \hat{B}u\}$$

$$y = Cx$$

$$\Rightarrow \quad y = \underbrace{CT^{-1}}_{\hat{C}}z$$

Similarity transform continued

Let the state feedback for system A, B be K.

$$\left\{\begin{matrix} \dot{x} = Ax + Bu \\ u = -Kx \end{matrix}\right\} \quad \Rightarrow \quad \dot{x} = \underbrace{(A - BK)}_{\Phi}x$$

In the transformed state, an equivalent control law is given by:

$$z = Tx; \quad x = T^{-1}z$$

$$\left\{\begin{matrix} \dot{z} = \hat{A}z + \hat{B}u \\ u = -Kx = -\underbrace{KT^{-1}}_{\hat{K}}z \end{matrix}\right\} \quad \Rightarrow \quad \dot{z} = \underbrace{(\hat{A} - \hat{B}\hat{K})}_{\hat{\Phi}}z$$

Compare controllability matrices

For the original system.

$$M_{cx}=[B,AB,A^2B,\cdots,A^{n-1}B]$$

For the transformed system.

$$M_{cz}=[\hat{B},\hat{A}\hat{B},\hat{A}^2\hat{B},\cdots,\hat{A}^{n-1}\hat{B}]$$

Controllability matrices are linked by the transformation matrix!

$$\hat{B}=TB,\quad \hat{A}=TAT^{-1}$$

$$M_{cz}=[TB,TAT^{-1}TB,(TAT^{-1})^2TB,\cdots,(TAT^{-1})^{n-1}TB]$$

$$M_{cz}=T[B,AB,A^2B,\cdots,A^{n-1}B]=TM_{cx}$$

REMARK

We can get the control canonical form directly from the transfer function so this is known for any A,B,C,D.

$$Y=[C(sI-A)^{-1}B+D]U$$

$$Y(s)=\frac{b_{n-1}s^{n-1}+\cdots+b_1s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_2s^2+a_1s+a_0}U(s)$$

$$\frac{d}{dt}[z]=\underbrace{\begin{bmatrix}-a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0\\ 1 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & 0\end{bmatrix}}_{\hat{A}}z+\underbrace{\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}}_{\hat{B}}u;$$

$$y=\underbrace{\begin{bmatrix}b_{n-1} & b_{n-2} & \cdots & b_0\end{bmatrix}}_{\hat{C}}z$$

INSIGHT

If we know the system in control canonical form, then we can find the transformation that takes us from an arbitrary form to the control canonical form!

Given form	Control canonical form
$\dot{x} = Ax + Bu$	$\dot{z} = \hat{A}z + \hat{B}u$

$$z = Tx; \quad x = T^{-1}z$$

$$\underbrace{[\hat{B}, \hat{A}\hat{B}, \hat{A}^2\hat{B}, \cdots, \hat{A}^{n-1}\hat{B}]}_{M_{cz}} = T\underbrace{[B, AB, A^2B, \cdots, A^{n-1}B]}_{M_{cx}}$$

INSIGHT CONTINUED

If we know the TRANSFORMATION that takes us from control canonical form to another form, we can do pole placement in control canonical form, and then find the equivalent state feedback for an alternative form.

Given form	Control canonical form
$\dot{x} = Ax + Bu; \quad u = -Kx$	$\dot{z} = \hat{A}z + \hat{B}u, \quad u = -\hat{K}z$

$$z = Tx; \quad x = T^{-1}z$$

$$M_{cz}[M_{cx}]^{-1} = T$$

$$\hat{K} = KT^{-1} \quad \Rightarrow \quad K = \hat{K}T$$

Pole placement algorithm

1. Find transfer function representation. $$C(sI-A)^{-1}B$$
2. Find control canonical form. $$\dot{z}=\hat{A}z+\hat{B}u$$
3. Find pole placement state feedback for control canonical form. $$u=-\hat{K}z$$
4. Find transformation matrix using controllability matrices. $$M_{cz}[M_{cx}]^{-1}=T$$
5. Find state feedback for original state space system. $$K=\hat{K}T$$

EXAMPLES

Example 1: Choose K to set the closed-loop poles at -1 and -2.

$$A=\begin{bmatrix}-1 & -2\\ 1 & -0.4\end{bmatrix};\quad B=\begin{bmatrix}1\\ -2\end{bmatrix};\quad C=\begin{bmatrix}3 & 4\end{bmatrix}$$

First find the control canonical form.

$$C(sI-A)^{-1}B=\frac{-5s+9.2}{s^2+1.4s+2.4}$$

$$\hat{A}=\begin{bmatrix}-1.4 & -2.4\\ 1 & 0\end{bmatrix};\quad \hat{B}=\begin{bmatrix}1\\ 0\end{bmatrix};\quad \hat{C}=\begin{bmatrix}-5 & 9.2\end{bmatrix}$$

Next find the similarity transform relating the two.

Example 1: Choose K to set the closed-loop poles at -1 and -2.

Find the controllability matrices.

$$A=\begin{bmatrix}-1 & -2\\ 1 & -0.4\end{bmatrix};\quad B=\begin{bmatrix}1\\ -2\end{bmatrix};$$

$$M_{cx}=[B, AB]$$

$$M_{cx}=\begin{bmatrix}-1 & 3\\ 2 & 1.8\end{bmatrix}$$

$$\hat{A}=\begin{bmatrix}-1.4 & -2.4\\ 1 & 0\end{bmatrix};\quad \hat{B}=\begin{bmatrix}1\\ 0\end{bmatrix};$$

$$M_{cz}=[\hat{B}, \hat{A}\hat{B}]$$

$$M_{cz}=\begin{bmatrix}1 & -1.4\\ 0 & 1\end{bmatrix}$$

Next find the similarity transform relating the two.

$$z=Tx \quad \Rightarrow \quad M_{cz}[M_{cx}]^{-1}=T$$

Example 1

Define feedback for control canonical form,
Desired pole polynomial is s^2+2s+1.

$$\hat{A}=\begin{bmatrix}-1.4 & -2.4\\ 1 & 0\end{bmatrix};\quad \hat{B}=\begin{bmatrix}1\\ 0\end{bmatrix};$$

$$\hat{K}=\begin{bmatrix}0.6 & -1.4\end{bmatrix}$$

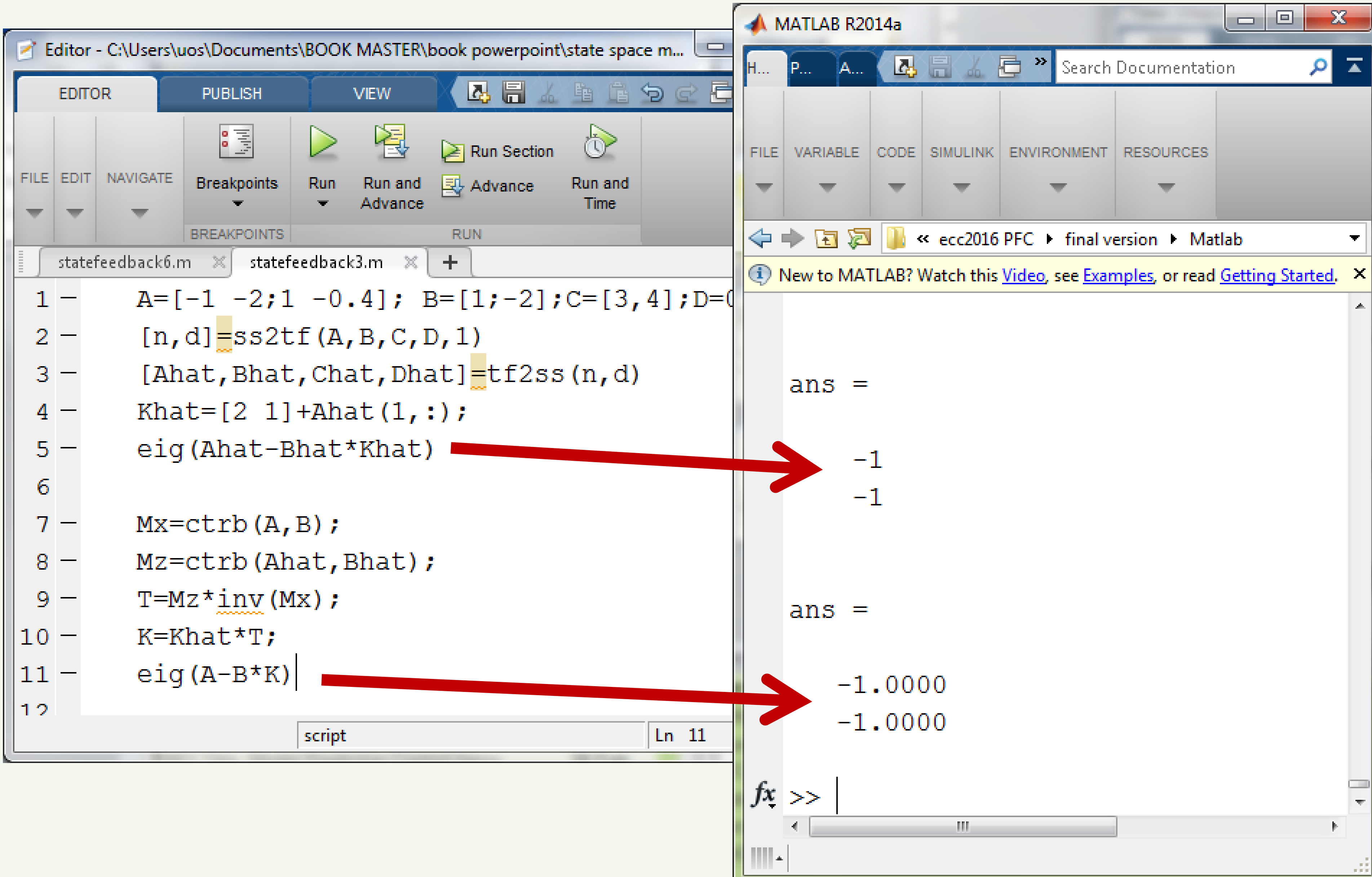
$$\hat{A}-\hat{B}\hat{K}=\begin{bmatrix}-1.4-k_1 & -2.4-k_2\\ 1 & 0\end{bmatrix}=\begin{bmatrix}-2 & -1\\ 1 & 0\end{bmatrix}$$

Find feedback for original system.

$$K=\hat{K}T;\quad M_{cz}[M_{cx}]^{-1}=T=\begin{bmatrix}-0.13 & -0.56\\ 0.26 & 0.13\end{bmatrix}$$

$$K=\begin{bmatrix}-0.44 & -0.52\end{bmatrix}$$

Confirm results using MATLAB



Example 2: Choose K to set the closed-loop poles at -0.5, -1 and -1.5.

$$A=\begin{bmatrix}-1 & -2 & -0.5\\ 0.2 & -0.4 & -0.6\\ 0 & -0.1 & 0.4\end{bmatrix};\quad B=\begin{bmatrix}1\\ -1\\ 0\end{bmatrix};\quad C=\begin{bmatrix}1 & 3 & 4\end{bmatrix}$$

First find the control canonical form.

$$C(sI-A)^{-1}B=\frac{-2s^2+1.2s+0.21}{s^3+s^2+0.18s-0.39}$$

$$\hat{A}=\begin{bmatrix}-1 & -0.18 & 0.39\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix};\quad \hat{B}=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix};\quad \hat{C}=\begin{bmatrix}-2 & 1.2 & 0.21\end{bmatrix}$$

Next find the similarity transform relating the two.

Example 2

Find the controllability matrices.

$$M_{cx} = [B, AB, A^2B]$$

$$M_{cx} = \begin{bmatrix} 1 & 1 & -2.25 \\ -1 & 0.6 & -0.1 \\ 0 & 0.1 & -0.02 \end{bmatrix}$$

$$M_{cz} = [\hat{B}, \hat{A}\hat{B}, \hat{A}^2\hat{B}]$$

$$M_{cz} = \begin{bmatrix} 1 & -1 & 0.82 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Next find the similarity transform relating the two.

$$z = Tx \quad \Rightarrow \quad M_{cz}[M_{cx}]^{-1} = T$$

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Example 2

Define feedback for control canonical form,
Desired pole polynomial is $s^3+3s^2+2.75s+0.75$.

$$\hat{A}-\hat{B}\hat{K} = \begin{bmatrix} -1-k_1 & -0.18-k_2 & 0.39-k_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & -2.75 & -0.75 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

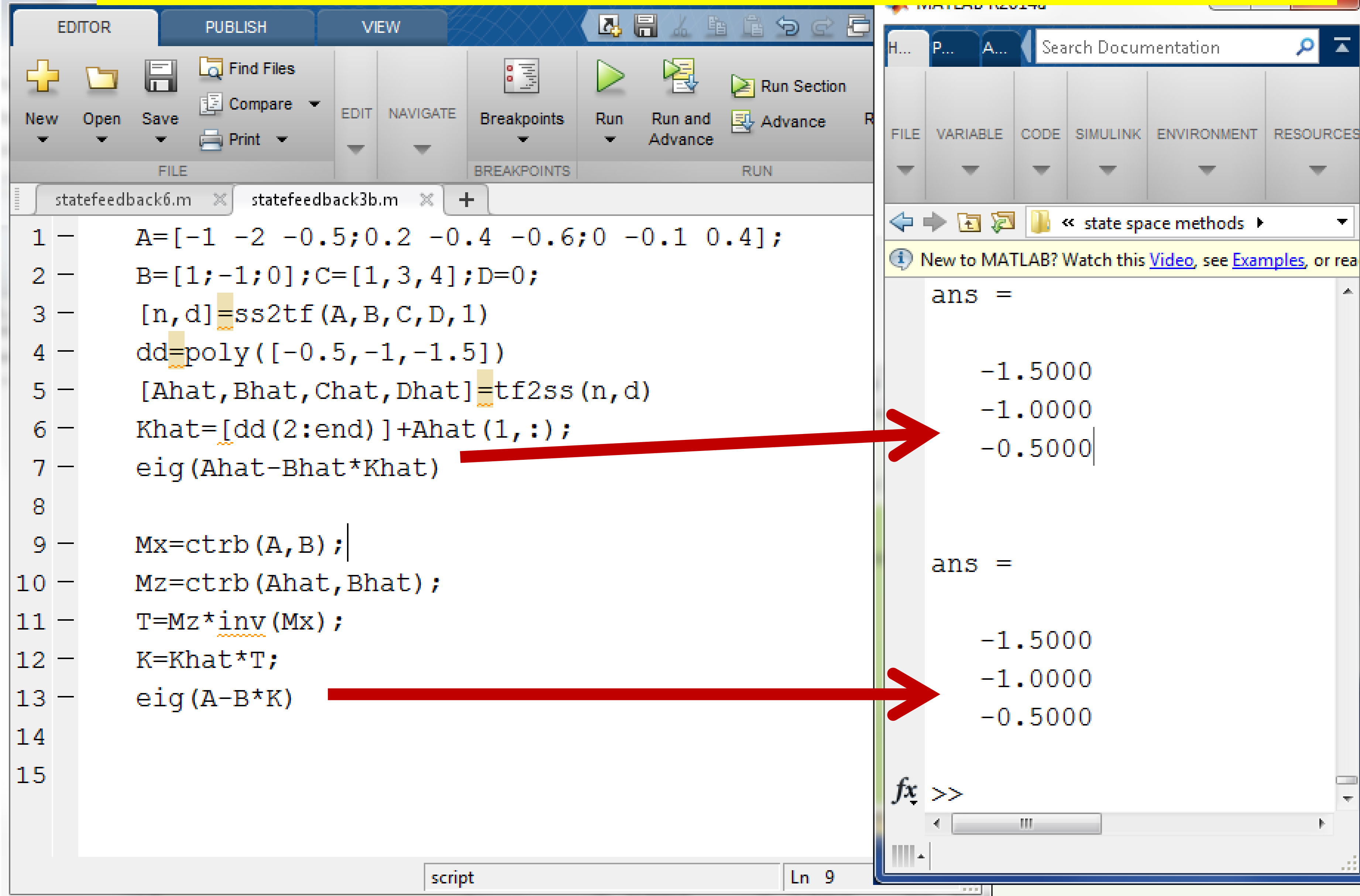
$$\hat{K} = \begin{bmatrix} 2 & 2.57 & 1.14 \end{bmatrix}$$

Find feedback for original system.

$$K = \hat{K}T; \quad M_{cz}[M_{cx}]^{-1} = T$$

$$K = \begin{bmatrix} -0.18 & -2.18 & 20.57 \end{bmatrix}$$

Confirm results using MATLAB



Summary $u = -Kx$

Introduced concepts of pole placement state feedback without a control canonical form.

1. Show that, assuming full controllability, there exists a transformation matrix to generate the equivalent control canonical form.
2. Pole placement design can be done using the canonical form.
3. Feedback parameters for the original states are obtained using the corresponding similarity transformation which is defined from the controllability matrices (see algorithm).
4. Not paper/pen exercise in general.

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