The University Of Sheffield.

State-space feedback 4 Ackermann's approach

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Introduction

- The previous videos showed how state feedback can place poles precisely as long as the system us fully controllable.

$$\left\{\begin{array}{c} \dot{x} = Ax + Bu \\ u = -Kx \end{array}\right\} \quad \Rightarrow \quad \dot{x} = \underbrace{(A - BK)}_{\Phi}x$$

- Both relied on control canonical forms.
- This video shows how one can find a suitable state feedback without resorting to canonical forms within the design procedure.

Remark

Much of this resource derives and explains the result.

If you only want to know the result, go directly towards the end.

Pole polynomials

The approach given here relies on clear definitions of two polynomials:

- the open-loop pole polynomial.

$$p_o = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_o$$

- the desired closed-loop pole polynomial.

$$p_c = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_o$$

Cayley-Hamilton Theorem

A matrix satisfies its own characteristic equation.

$$|\lambda I - A| = 0 = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0$$

This implies that.

$$A^n + a_{n-1}A^{n-1} + \cdots + a_0 A^0 = 0$$

The proof is omitted but relatively straightforward and available in most text books.

Corollary

The following 2 identities are known to be true.

$$p_o = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_o$$

$$A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I = 0$$

$$p_c = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_o$$

$$(A-BK)^n + \alpha_{n-1}(A-BK)^{n-1} + \cdots + \alpha_1(A-BK) + \alpha_0 I = 0$$

Finding K such that this last identify is satisfied implies achieving target poles.

Ackermann's approach

The intention is to exploit the latter of these identities and full rank nature of the controllability matrix.

First define:

$$p_c(A-BK)=(A-BK)^n+\alpha_{n-1}(A-BK)^{n-1}+\cdots+\alpha_1(A-BK)+\alpha_0 I$$

$$p_c(A)=(A)^n+\cdots+\alpha_1(A)+\alpha_0 I$$

The first concept is to show that $p_c(A)$ is a component of $p_c(A\text{-}BK)$ and exploit this in the algebra.

NOTE, by definition:

$$p_c(A-BK)=0$$

$$p_c(A)\neq 0$$

Ackermann's approach

The first concept is to expand $p_c(A\text{-}BK)$ and unpack the underlying structure. First consider each term.

$$(A-BK)^n = A^n - A^{n-1}BK + \cdots + (-1)^{n-1}A(BK)^{n-1} + (-1)^n Bf_n(A,BK)$$
$$(A-BK)^{n-1} = A^{n-1} - A^{n-2}BK + \cdots + (-1)^{n-2}A(BK)^{n-2} + (-1)^{n-1} Bf_{n-1}(A,BK)$$
$$(A-BK)^{n-2} = A^{n-2} - A^{n-3}BK + \cdots + (-1)^{n-3}A(BK)^{n-3} + (-1)^{n-2} Bf_{n-2}(A,BK)$$
$$\vdots$$
$$(A-BK)^2 = A^2 - ABK - BKA + (BK)^2$$
$$A-BK = A-BK$$

Deliberate separation of terms with a leading 'A' and a leading 'B'.

Ackermann's approach

The next step is to show that one can express each term using the controllability matrix.

$$(A-BK)^n = A^n - A^{n-1}BK + \cdots + (-1)^{n-1}A(BK)^{n-1} + (-1)^n Bf_n(A,BK)$$

$$(A-BK)^n = A^n + \underbrace{[B, AB, \cdots, A^{n-2}B, A^{n-1}B]}_{M_c}\underbrace{\begin{bmatrix} (-1)^n f_n(A,BK) \\ (-1)^{n-1}K(BK)^{n-2} \\ \vdots \\ -K \end{bmatrix}}_{v_n}$$

$$(A-BK)^3 = A^3 - A^2BK + A(BK)^2 - Bf_3(A,BK)$$

$$(A-BK)^3 = A^3 + \underbrace{[B, AB, A^2B, \cdots, A^{n-1}B]}_{M_c}\underbrace{\begin{bmatrix} -f_3(A,BK) \\ K(BK) \\ -K \\ \vdots \\ 0 \end{bmatrix}}_{v_3}$$

The ZERO is key

Ackermann's approach

Next expand p_c(A-BK) and unpack the underlying structure. First consider each term.

$$
\begin{aligned}
(A-BK)^n &= A^n + M_c v_n \\
(A-BK)^{n-1} &= A^{n-1} + M_c v_{n-1} \\
&\vdots \\
(A-BK)^2 &= A^2 + M_c v_2 \\
A-BK &= A + M_c v_1
\end{aligned}
$$

Only v_n is non-zero in the bottom row!

$$p_c(A-BK) = (A-BK)^n + \alpha_{n-1}(A-BK)^{n-1} + \cdots + \alpha_1(A-BK) + \alpha_0 I$$

$$
\begin{aligned}
p_c(A-BK) = {} & A^n + \alpha_{n-1}A^{n-1} + \cdots + \alpha_1 A + \alpha_0 I \\
& + M_c[v_n + \alpha_{n-1}v_{n-1} + \cdots + \alpha_1 v_1]
\end{aligned}
$$

Ackermann's approach

Show that $p_c(A)$ is a component of $p_c(A\text{-}BK)$ and exploit this in the algebra.

$$p_c(A-BK)=A^n+\alpha_{n-1}A^{n-1}+\cdots+\alpha_1 A+\alpha_0 I$$
$$+M_c[v_n+\alpha_{n-1}v_{n-1}+\cdots+\alpha_1 v_1]$$

$$p_c(A-BK)=0$$
$$p_c(A)\neq 0$$

$$0=p_c(A)+M_c[v_n+\alpha_{n-1}v_{n-1}+\cdots+\alpha_1 v_1]$$

$$M_c^{-1}p_c(A)=-[v_n+\alpha_{n-1}v_{n-1}+\cdots+\alpha_1 v_1]$$

Ackermann's approach

Exploit the structure of v_n by extracting only the last row of each side of the equation and thus extracting the feedback gain K.

$$\underbrace{\begin{bmatrix} (-1)^n f_n(A, BK) \\ (-1)^{n-1} K(BK)^{n-2} \\ \vdots \\ -K \end{bmatrix}}_{v_n}$$

$$M_c^{-1} p_c(A) = -[v_n + \alpha_{n-1} v_{n-1} + \cdots + \alpha_1 v_1]$$

$$\begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix} v_{n-1} = 0, \quad \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix} v_{n-2} = 0, \quad \cdots$$

$$\begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} M_c^{-1} p_c(A) = K$$

As p_c is by definition, one can find the required state feedback using this formulae.

Summary of Ackermann's approach

Define the desired closed-loop pole polynomial.

$$p_c = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_o$$

The required feedback is given from the formula where M_c is the controllability matrix.

$$\begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix} M_c^{-1} p_c(A) = K$$

EXAMPLES

Example 1: Choose K to set the closed-loop poles at -1 and -2.

$$A=\begin{bmatrix}-1 & -2\\ 1 & -0.4\end{bmatrix};\quad B=\begin{bmatrix}1\\ -2\end{bmatrix};\quad C=\begin{bmatrix}3 & 4\end{bmatrix}$$

First find the required pole polynomial

$$p_c=s^2+3s+2$$

Find $p_c(A)$

$$p_c(A)=\begin{bmatrix}-1 & -2\\ 1 & -0.4\end{bmatrix}^2+3\begin{bmatrix}-1 & -2\\ 1 & -0.4\end{bmatrix}+\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}=\begin{bmatrix}-2 & -3.2\\ 1.6 & -1.04\end{bmatrix}$$

Example 1: Choose K to set the closed-loop poles at -1 and -2.

$$A=\begin{bmatrix}-1 & -2\\ 1 & -0.4\end{bmatrix};\quad B=\begin{bmatrix}1\\ -2\end{bmatrix};\qquad p_c(A)=\begin{bmatrix}-2 & -3.2\\ 1.6 & -1.04\end{bmatrix}$$

Define the controllability matrix

$$M_c=[B,AB]=\begin{bmatrix}1 & 3\\ -2 & 1.8\end{bmatrix}$$

Use Ackermann's formulae

$$\begin{bmatrix}0 & \cdots & 0 & 1\end{bmatrix}M_c^{-1}p_c(A)=K=\begin{bmatrix}-0.31 & -0.95\end{bmatrix}$$

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statefeedback3b.m | statefeedback4.m*

```matlab
A=[-1 -2;1 -0.4]; B=[1;-2];
C=[3,4];D=0;
Mx=ctrb(A,B)
PcA=A^2+3*A+2*eye(2)
K=[0 1]*inv(Mx)*PcA
eig(A-B*K)
```

script | Ln 8 Col 1

MATLAB R2014a

book powerpoint ▸ state space methods ▸

New to MATLAB? Watch this Video, see Examples, or read Getting Starte

```
Mx =

    1.0000    3.0000
   -2.0000    1.8000


PcA =

   -2.0000   -3.2000
    1.6000   -1.0400


K =

   -0.3077   -0.9538
```

Example 2: Choose K to set the closed-loop poles at -0.5, -1 and -1.5.

$$A=\begin{bmatrix}-1 & -2 & -0.5\\ 0.2 & -0.4 & -0.6\\ 0 & -0.1 & 0.4\end{bmatrix};\quad B=\begin{bmatrix}1\\ -1\\ 0\end{bmatrix};\quad C=\begin{bmatrix}1 & 3 & 4\end{bmatrix}$$

First find the required pole polynomial

$$p_c = s^3 + 3s^2 + 2.75s + 0.75$$

Find $p_c(A)$ [Use MATLAB as tedious]

$$p_c(A) = A^3 + 3A^2 + 2.75A + 0.75I = \begin{bmatrix}-0.23 & 0.56 & 1.715\\ -0.046 & -0.248 & -1.742\\ -0.04 & -0.257 & 2.608\end{bmatrix}$$

Example 2: Choose K to set the closed-loop poles at -1 and -2.

$$A = \begin{bmatrix} -1 & -2 & -0.5 \\ 0.2 & -0.4 & -0.6 \\ 0 & -0.1 & 0.4 \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix};$$

$$p_c(A) = \begin{bmatrix} -0.23 & 0.56 & 1.715 \\ -0.046 & -0.248 & -1.742 \\ -0.04 & -0.257 & 2.608 \end{bmatrix}$$

Define the controllability matrix

$$M_c = [B, AB, A^2B] = \begin{bmatrix} 1 & 1 & -2.25 \\ -1 & 0.6 & -0.1 \\ 0 & 0.1 & -0.02 \end{bmatrix}$$

Use Ackermann's formulae

$$[0 \quad \cdots \quad 0 \quad 1] M_c^{-1} p_c(A) = K = [-0.18 \quad -2.18 \quad 20.57]$$

Summary $u = -Kx$

Introduced concepts of pole placement state feedback without a control canonical form.

1. Show that, assuming full controllability, one can use Ackermann's formulae to define the required feedback to achieve any desired pole polynomial.

$$\boxed{\begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix} M_c^{-1} p_c(A) = K}$$

2. Not paper/pen exercise in general as involves substantial matrix multiplication and inversion.
3. Numerically poorly conditioned for weakly controllable and large dimension systems.

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