State-space feedback 5
Tutorial examples and use of MATLAB

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Introduction

- The previous videos showed how state feedback can place poles precisely as long as the system us fully controllable.

$$\left\{\begin{array}{c} \dot{x} = Ax + Bu \\ u = -Kx \end{array}\right\} \quad \Rightarrow \quad \dot{x} = \underbrace{(A - BK)}_{\Phi}x$$

- This video illustrates and compares the three different methods presented.
  1. Using canonical forms.
  2. Using a transformation to obtain a canonical form.
  3. Ackermann's method.

Remarks

1. In general, pole placement for state space models is not a paper and pen exercise.
2. With the exception of 2 by 2 systems, the required algebra is tedious and students should use software once they are comfortable with the key principles. MATLAB use is demonstrated.
3. All the methods assume controllability and that the user can specify, in advance, the desired closed-loop pole polynomial.

$$p_c = s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_1 s + \alpha_o$$

Pole placement algorithm

1. Find transfer function representation.
   $$C(sI-A)^{-1}B$$
2. Find control canonical form.
   $$\dot{z}=\hat{A}z+\hat{B}u$$
3. Find pole placement state feedback for control canonical form.
   $$u=-\hat{K}z$$
4. Find transformation matrix using controllability matrices.
   $$M_{cz}[M_{cx}]^{-1}=T$$
5. Find state feedback for original state space system.
   $$K=\hat{K}T$$

Ackermann's approach

Define M_c to be the controllability matrix, then:

$$[0 \quad \cdots \quad 0 \quad 1] M_c^{-1} p_c(A) = K$$

EXAMPLES

Example 1:

Choose K to give closed-loop poles at -1±j0.5.

$$A=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix};\quad B=\begin{bmatrix}0\\ 1\end{bmatrix};$$

First find the required pole polynomial

$$p_c = s^2 + 2s + 1.25$$

The system is not in control canonical form so that approach is not viable.

Try Ackermann and state transformation.

Example 1 - Ackermann

$$\begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} M_c^{-1} p_c(A) = K$$

State transformation

1. Find control canonical form. Open loop pole polynomial is $p_o=s^2$.

$$p_c = s^2 + 2s + 1.25$$

2. Find pole placement state feedback for control canonical form.

$$u = -\hat{K}z$$

$$\hat{A} - \hat{B}\hat{K} = \begin{bmatrix} -2 & -1.25 \\ 1 & 0 \end{bmatrix} =$$

State transformation

Find transformation matrix using controllability matrices.

$$M_{cz}[M_{cx}]^{-1} = T$$

Find state feedback for original state space system.

$$K = \hat{K}T$$

MATLAB CODE

Ackermann

```
Mx=ctrb(A,B)
pcA=A^2+2*A+1.25*eye(2)
K=[0 1]*inv(Mx)*pcA
eig(A-B*K)
```

Transformation

```
[Ahat,Bhat,Chat,Dhat]=tf2ss(1,[1 0 0])
Khat=[2 1.25]+Ahat(1,:)
eig(Ahat-Bhat*Khat)
Mz=[Bhat,Ahat*Bhat]
T=Mz*inv(Mx)
K=Khat*T
```

Example 2:

Choose K to set the closed-loop poles all at -2.

$$A = \begin{bmatrix} -6 & -11 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

First find the required closed-loop pole polynomial

$$p_c = (s+2)^3 = s^3 + 6s^2 + 12s + 8$$

Canonical approach - example 2

$$p_c = s^3 + 6s^2 + 12s + 8$$

$$A - BK = A_c = \begin{bmatrix} -6 & -12 & -8 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Desired closed-loop A matrix

Transformation approach – example 2

Already in canonical form so not needed!

Example 2 - Ackermann

$$\begin{bmatrix}0 & \cdots & 0 & 1\end{bmatrix} M_c^{-1} p_c(A) = K$$

$$p_c(A) = A^3 + 6A^2 + 12A + 8I = \begin{bmatrix} -4 & -11 & -6 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

Not paper/pen exercise in general!

$$M_c = [B, AB, A^2B] = \begin{bmatrix} 1 & -6 & 25 \\ 0 & -1 & -6 \\ 0 & 0 & 1 \end{bmatrix}$$

$$K = \begin{bmatrix}0 & 1 & 2\end{bmatrix}$$

MATLAB CODE

Ackermann

```
dd=[1 6 12 8];
Mx=ctrb(A,B)
PcA=A^3+dd(2)*A^2+dd(3)*A+dd(4)*eye(3)
K=[0 0 1]*inv(Mx)*PcA
eig(A-B*K)
```

Example 3:

Choose K to set the closed-loop poles at –1,-1,-1,-2.

$$A=\begin{bmatrix}-5.47 & -0.98 & -4 & -1\\ 10.52 & -0.03 & 6.96 & 1.74\\ 0.864 & 0.5 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix};\quad B=\begin{bmatrix}0.5\\ -0.87\\ 0\\ 0\end{bmatrix}$$

First find the required closed-loop pole polynomial but also note that one cannot reasonably tackle this question on pen and paper.

$$p_c = s^4 + 5s^3 + 9s^2 + 7s + 2$$

Canonical approach - example 3

System not in canonical form so cannot use this approach.

State transformation

1. Find control canonical form. Open loop pole polynomial is:

$$p_o = s^4 + 5.5s^3 + 10.5s^2 + 8s + 2$$

$$\hat{A} = \begin{bmatrix} -5.5 & -10.5 & -8 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}; \quad \hat{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$p_c = s^4 + 5s^3 + 9s^2 + 7s + 2$$

$$\hat{A} - \hat{B}\hat{K} = \begin{bmatrix} -5 & -9 & -7 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -5.5-k_1 & -10.5-k_2 & -8-k_3 & -2-k_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\hat{K} = \begin{bmatrix} -0.5 & -1.5 & -1 & 0 \end{bmatrix}$$

State transformation

Find transformation matrix using controllability matrices.

$$M_{cz}[M_{cx}]^{-1} = T$$

Find state feedback for original state space system.

$$K = \hat{K}T$$

$$K = \begin{bmatrix} -1.54 & -0.31 & -1 & 0 \end{bmatrix}$$

The University Of Sheffield.

Example 3 - Ackermann

$$\begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} M_c^{-1} p_c(A) = K$$

Not paper/pen exercise in general!

$$p_c(A) = A^4 + 5A^3 + 9A^2 + 7A + 2I$$

$$M_c = [B, AB, A^2B, A^3B]$$

$$K = \begin{bmatrix} -1.54 & -0.31 & -1 & 0 \end{bmatrix}$$

MATLAB CODE

Ackermann

```
Mx=ctrb(A,B)
dd=poly([-1,-1,-1,-2]);
PcA=A^4+dd(2)*A^3+dd(3)*A
^2+dd(4)*A+dd(5)*eye(4)
K=[0 0 0 1]*inv(Mx)*PcA
eig(A-B*K)
```

Transformation

```
dd=poly([-1,-1,-1,-2]);
d=poly([-0.5,-1,-2,-2]);
[Ahat,Bhat,Chat]=tf2ss(1,d);
Khat=[dd(2:end)]+Ahat(1,:)
Mz=ctrb(Ahat,Bhat)
Mx=ctrb(A,B)
T=Mz*inv(Mx);
K=Khat*T
eig(A-B*K)
```

MATLAB shortcuts

MATLAB has built in tools to find K in a single line rather than the detailed code illustrate earlier.

place.m
K = place(A,B,poles)

acker.m
K = acker(A,B,poles)

Both are demonstrated now on the 3 examples given earlier.

place.m does not do multiple poles with a single input.

acker.m is sensitive for large numbers of states/poor controllability

Summary

1. Given some worked examples of pole placement designs.
2. Shown that obtain the same answer with 3 different methods.
3. Also made it clear that in general, these designs are not paper and pen exercises and students should use a computer for the algebra.
4. Demonstrated use of MATLAB tools.

Open Engineering Resources

HEA Engineering Subject Centre
A HEFCE/JISC/HEA Pilot Project

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