State-space feedback 7 optimal control

J A Rossiter

Slides by Anthony Rossiter

Introduction

- The earlier videos introduced the concept of state feedback and demonstrated that it moves the poles.

$$\left\{\begin{matrix} \dot{x} = Ax + Bu \\ u = -Kx \end{matrix}\right\} \quad \Rightarrow \quad \dot{x} = \underbrace{(A - BK)}_{\Phi} x$$

- It was shown that when a system is fully controllable, the poles can be placed arbitrarily, that is wherever the user desires.
- **However, it is clear that in general, it is not obvious where to place the poles and hence a more systematic design procedure is required.**

Impact of pole positions

Typical guidance, for example arising from a root-loci analysis, would suggest that closed-loop poles should be placed near to open-loop poles to avoid aggressive inputs and/or loop sensitivity.

Nevertheless, close is ill-defined and also, having to place n poles explicitly rather than just a nominal area for dominant poles results in an over-exacting or overly specified design.

In practise we are happy for poles to be in a 'specified region', we do not need to specify them precisely.

Proposal

- Instead of specifying pole positions precisely, we may have more effective degrees of freedom if we focus on a proxy for closed-loop performance.
- Such a proxy should be such that it can allow systematic design while giving enough slack for the feedback to place non-dominant poles where ever it likes, thus not using up degrees of freedom unnecessarily.
- Such a proxy is often a performance index.

Performance index

A performance index **J** is a mathematical measure of the quality of system behaviour.

Large J implies poor performance and small J implies good performance.

Performance is often defined using attributes such as:

Others are possible but not discussed as not mainstream.

1. Rise time
2. Settling time
3. Overshoot, oscillation and damping ratios.
4. Offset.
5. Peak values of signals (especially inputs).

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Common performance index

A typical performance index is a quadratic measure of future behaviour (using the origin as the target) and hence:

$$J = \int_0^{\infty} (x^T Q x + u^T R u) \quad dt$$

Weighted squares of deviation of states from target.

Weighted squares of control activity.

NOTE: To some extent this is an arbitrary choice.

Squares are used because these lead to easier analysis and well behaved solutions which are relatively insensitive to changes in initial conditions.

Performance index analysis

The selected performance index allows for relatively systematic design.

$$J = \int_0^\infty (x^T Q x + u^T R u) \quad dt$$

This term implicitly measures convergence rate (rise time and settling).

This term penalises aggressive use of the input.

The choices of Q and R allow trade offs between input activity and rates of convergence.

Performance index

$$J=\int_0^{\infty}(x^TQx+u^TRu)\quad dt$$

Performance is often defined using attributes:

1. Rise time implicit in x^TQx term.
2. Settling time implicit in x^TQx term.
3. Overshoot, oscillation and damping ratios implicit due to use of squares which penalise +ve and –ve errors equally.
4. Offset implicit in use of infinite horizons (J bounded only if errors go to zero).
5. Peak values of signals (including input) penalised due to use of squares.

Key performance characteristics are implicit.

Optimal control design

How do we optimise the performance index with respect to the parameters of a state feedback and subject to the given dynamics?

$$\underbrace{\min}_{K} J = \int_0^\infty (x^T Q x + u^T R u) \quad dt$$

$$\begin{Bmatrix} \dot{x} = Ax + Bu \\ u = -Kx \end{Bmatrix} \Rightarrow \quad \dot{x} = \underbrace{(A - BK)}_{\Phi} x$$

The is done via dynamic programming. Derivation excluded here for brevity and solution only provided.

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Optimal control solution

Optimal state feedback is given as follows.

$$\underbrace{\min}_{K} J = \int_0^{\infty} (x^T Q x + u^T R u) \quad dt \quad s.t. \quad \left\{ \begin{matrix} \dot{x} = Ax + Bu \\ u = -Kx \end{matrix} \right\}$$

$$K = R^{-1} B^T P$$

$$A^T P + PA - PBR^{-1} B^T P + Q = 0$$

P is symmetric.

Require that R is invertible (logical as otherwise no weighting on input activity).

Remarks

1. Assuming controllability, optimal state feedback is guaranteed to be stabilising. This follows easily from dynamic programming or otherwise.
2. The required computations are not amenable to pen and paper in general; students are recommended to use a computer.
3. MATLAB has a command: **K=lqr(A,B,Q,R)**
4. Common to choose Q➔C^TQC so that, in effect:

$$\underbrace{\min}_{K} J = \int_0^{\infty} (y^T \hat{Q} y + u^T R u) \quad dt; \quad Q = C^T \hat{Q} C$$

Optimal state feedback for discrete systems

This is analogous to continuous time, but uses a sum to infinity rather than an integral.

$$\underbrace{\min}_{K} J = \sum_{k=0}^{\infty} (x_{k+1}^T Q x_k + u_k^T R u_k)$$

$$\left\{ \begin{array}{c} x_{k+1} = Ax_k + Bu_k \\ u_k = -Kx_k \end{array} \right\}$$

The solution is derived from (**or use dlqr.m**):

$$K = [R + B^T PB]^{-1} B^T PA$$

$$P = Q + A^T PA - A^T PB[R + B^T PB]^{-1} B^T PA$$

NUMERICAL EXAMPLES

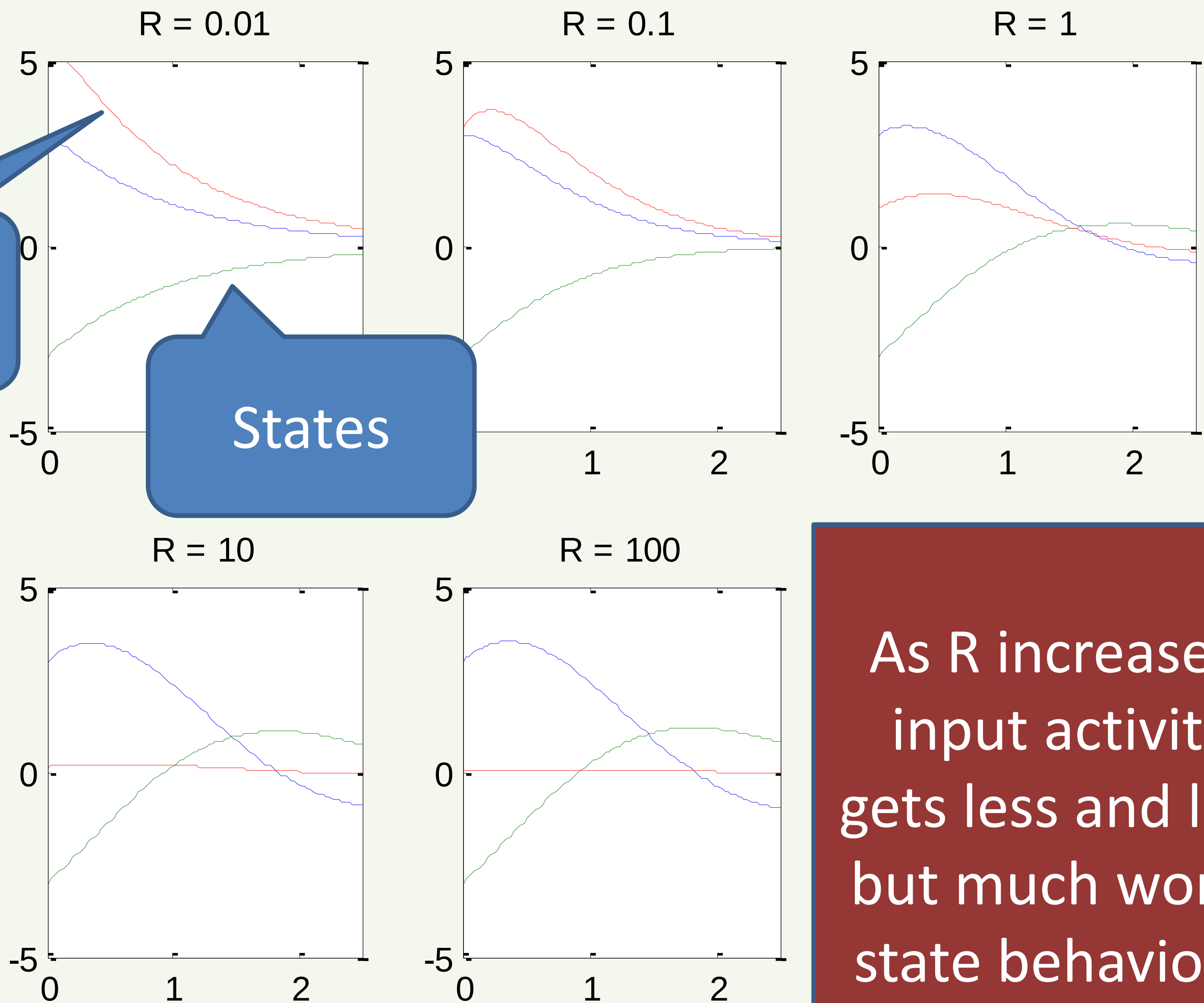
Examples

Compare the closed-loop state behaviour with different choices of R.

$$A=\begin{bmatrix}-1 & -2\\ 1 & 0\end{bmatrix};\quad B=\begin{bmatrix}1\\0\end{bmatrix};\quad Q=I$$

$$A=\begin{bmatrix}-6 & -11 & -6\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix};\quad B=\begin{bmatrix}1\\0\\0\end{bmatrix};\quad Q=I$$

$$\underbrace{\min}_{K} J=\int_0^\infty (x^TQx+u^TRu)\quad dt$$



As R increases, input activity gets less and less but much worse state behaviour.

Example 1

Interim remarks

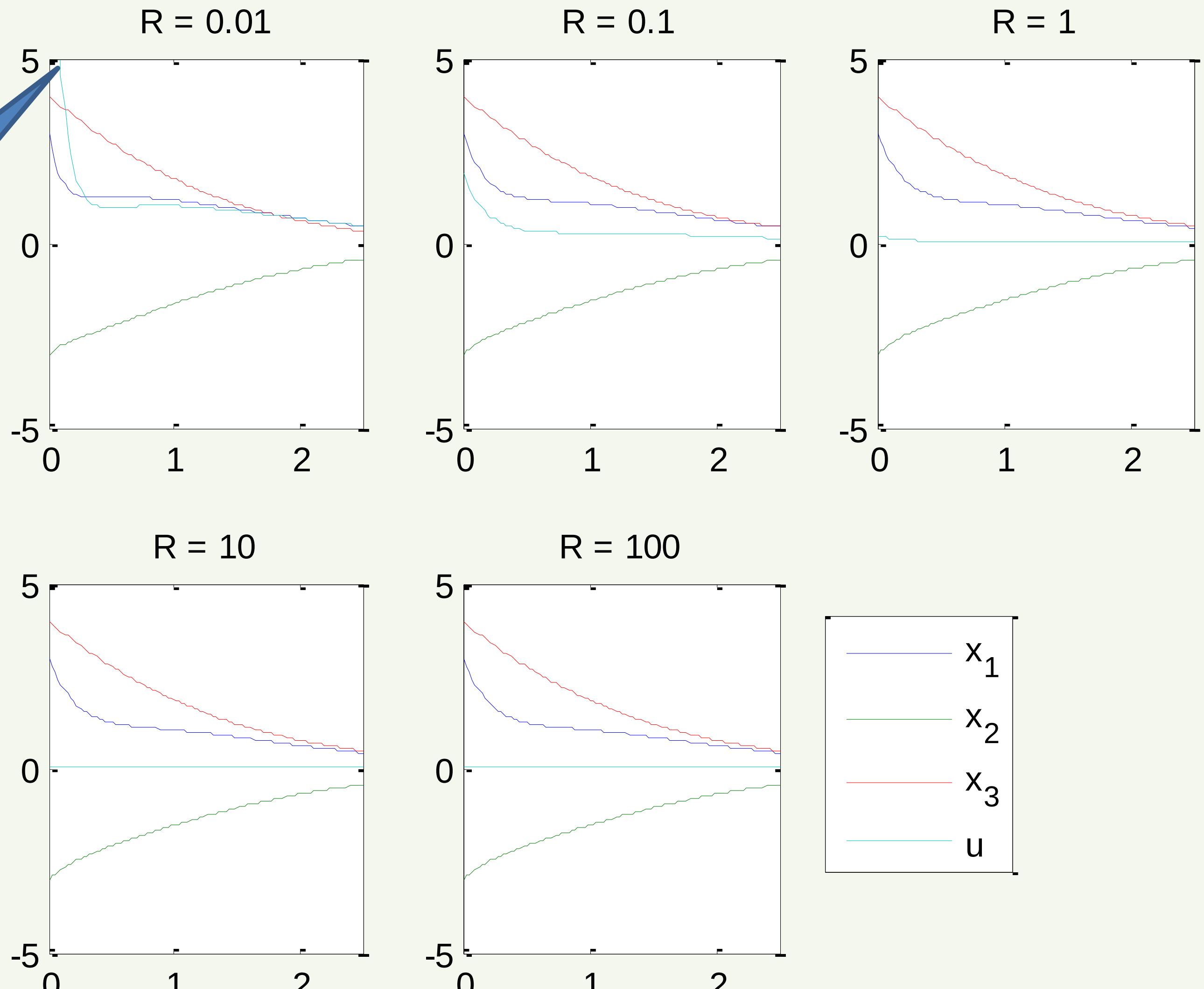
If R is selected to be to big, then the input activity term dominates J and one may not get good state behaviour, especially where open-loop dynamics are poor.

$$\underbrace{\min}_{K} J = \int_0^\infty (x^T Q x + u^T R u) \quad dt$$

If R is too small, the focus is all on state convergence, with no regard to the input activity required.

Input signal

Example 2



As R increases, input activity gets less and less but slower state behaviour.

System has poor controllability – some modes remain slow.

Summary $u = -Kx$

1. When a system is in controllable form, every coefficient of the closed-loop pole polynomial can be defined as desired using state feedback.
2. Optimal state feedback gives a systematic approach to pole selection in that it focuses on closed-loop performance, and thus selects pole positions only indirectly.
3. In general, optimal state feedback gives better behaviour than pole placement by enabling more systematic tuning/trade-offs between tracking and control activity.
4. However, if the system has poor controllability, some modes will be largely unaffected.

Anthony Rossiter
Department of Automatic Control and Systems Engineering
University of Sheffield
www.shef.ac.uk/acse