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State-space observers 3
basic design

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Introduction

The previous video derived the basic structure of a state observer.

1. Simulate a process model in parallel, but modify the dynamics with an error term linked to the measured system and model outputs.

$$\left\{\begin{array}{c} \dot{x} = Ax + Bu \\ y = Cx \end{array}\right\}$$

$$\left\{\begin{array}{c} \dot{z} = Az + Bu + L(y - y_m) \\ y_m = Cz \end{array}\right\}$$

2. It is straightforward to show that the error state is governed by.

$$\dot{x} - \dot{z} = (A - LC)(x - z)$$

Remarks

An observer has a simple structure.

$$\left\{ \begin{matrix} \dot{z} = Az + Bu + L(y - y_m) \\ y_m = Cz \end{matrix} \right\}$$

The key design parameter is the matrix L which is chosen to determine the dynamics of the state measurement error.

$$\dot{x} - \dot{z} = (A - LC)(x - z)$$

Context

The aim is to give the state measurement error dynamics the desirable dynamic.

Ideal the state errors would converge quickly and be small.

$$\dot{x} - \dot{z} = (A - LC)(x - z)$$

A simple aim then would be to place the poles of the error dynamics as desired.

$$\lambda(A - LC) = p_{1,} p_2, \cdots$$

How do we place these poles?

Duality with state feedback

State feedback design focussed on placing the poles of the system

$$\begin{Bmatrix} \dot{x} = Ax + Bu \\ u = -Kx \end{Bmatrix} \quad \Rightarrow \quad \dot{x} = (A - BK)x$$

We know several algorithms for choosing K to place the eigenvalues of A-BK

The observer problem looks very similar.

$$\dot{e} = (A - LC)e$$

Duality with state feedback

Choose K to place the eigenvalues of A-BK

Choose L to place the eigenvalues of A-LC

$$\dot{x} = (A - BK)x$$

$$\dot{e} = (A - LC)e$$

Eigenvalues of a transpose are the same.

$$\left|\lambda I - (A - LC)\right| = 0$$

$$\left|\lambda I - (A - LC)^T\right| = 0$$

$$\left|\lambda I - (A^T - C^T L^T)\right| = 0 \quad \equiv \quad \left|\lambda I - (A - BK)\right| = 0$$

Duality with state feedback

Choose K to place the eigenvalues of A-BK

Choose L to place the eigenvalues of A^T-C^TL^T

$$\left|\lambda I - (A^T - C^T L^T)\right| = 0 \quad \equiv \quad \left|\lambda I - (A - BK)\right| = 0$$

C^T is equivalent to B

L^T is equivalent to K

Interim summary

Any pole placement algorithms used to choose a state feedback K to place the poles of A-BK can be used to choose L^T to place the poles of A^T-C^TL^T.

CAVIAT

1. Pole placement relied on controllability, that is the M_c matrix is full rank.

$$M_c = [B, AB, A^2B, \cdots, A^{n-1}B]$$

2. Pole placement on A^T-C^TL^T requires M_o to be full rank.

$$M_o = [C^T, A^TC^T, \cdots, (A^T)^{n-1}C^T]$$

Remark

Pole placement on $A^T-C^TL^T$ requires M_o to be full rank.

$$M_o = [C^T, A^T C^T, \cdots, (A^T)^{n-1} C^T]$$

Viewers will notice immediately that this is the transpose of the observability matrix.

SUMMARY: The poles of the observer can be placed arbitrarily if and only if the system has full observability.

Placing poles for an observer

We can deploy any of the algorithms discussed on state feedback.

1) Using canonical forms.
2) Ackermann's formulae.
3) Using state transformation.

These will be summarised briefly for convenience.

Summary of Ackermann's approach

Define the desired closed-loop pole polynomial.

$$p_c = s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_1 s + \alpha_o$$

The required observer gain is given from the formula where M_0 is the observability matrix.

NOTE A^T

$$[0 \quad 0 \quad \cdots \quad 1] M_o^{-1} p_c(A^T) = L^T$$

$$M_o = [C^T, A^T C^T, \cdots, (A^T)^{n-1} C^T]$$

Impact of state feedback on a canonical form.

One can now form the closed-loop state space model by inspection.

$$\left\{\begin{array}{l} \dot{x} = Ax + Bu \\ u = -Kx \end{array}\right\} \quad \Rightarrow \quad \dot{x} = \underbrace{(A - BK)}_{\Phi} x$$

$$A - BK = \begin{bmatrix} -a_{n-1} - k_1 & -a_{n-2} - k_2 & \cdots & -a_1 - k_{n-1} & -a_0 - k_n \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

$$\left|\lambda I - A + BK\right| = \lambda^n + (a_{n-1} + k_1)\lambda^{n-1} + \cdots + (a_0 + k_n)$$

One can choose the parameters of the closed-loop pole polynomial directly by choosing the parameters k_i.

Impact of observer gain on a canonical form.

One can now form the closed-loop state space model by inspection.

$$\dot{e} = (A^T - C^T L^T)e$$

$$A^T - C^T L^T = \begin{bmatrix} -a_{n-1} - l_1 & -a_{n-2} - l_2 & \cdots & -a_1 - l_{n-1} & -a_0 - l_n \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

$$|\lambda I - A + LC| = \lambda^n + (a_{n-1} + l_1)\lambda^{n-1} + \cdots + (a_0 + l_n)$$

Note this formulae is full of transposes.

Observer gain with a canonical form.

$$A^T - C^T L^T = \begin{bmatrix} -a_{n-1} - l_1 & -a_{n-2} - l_2 & \cdots & -a_1 - l_{n-1} & -a_0 - l_n \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

$$A - LC = \begin{bmatrix} -a_{n-1} - l_1 & 1 & \cdots & 0 \\ -a_{n-2} - l_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ -a_0 - l_n & 0 & \cdots & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} -a_{n-1} & 1 & \cdots & 0 \\ -a_{n-2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ -a_0 & 0 & \cdots & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_n \end{bmatrix}; \quad C = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$$

Observer canonical form

Using MATLAB

As with state feedback, in practice numerical computation of observer gains is not a paper and pen exercise and hence use of software is recommended.

This is almost identical to state feedback design with the exception of using transposes where appropriate.

Some quick illustrations are given next.

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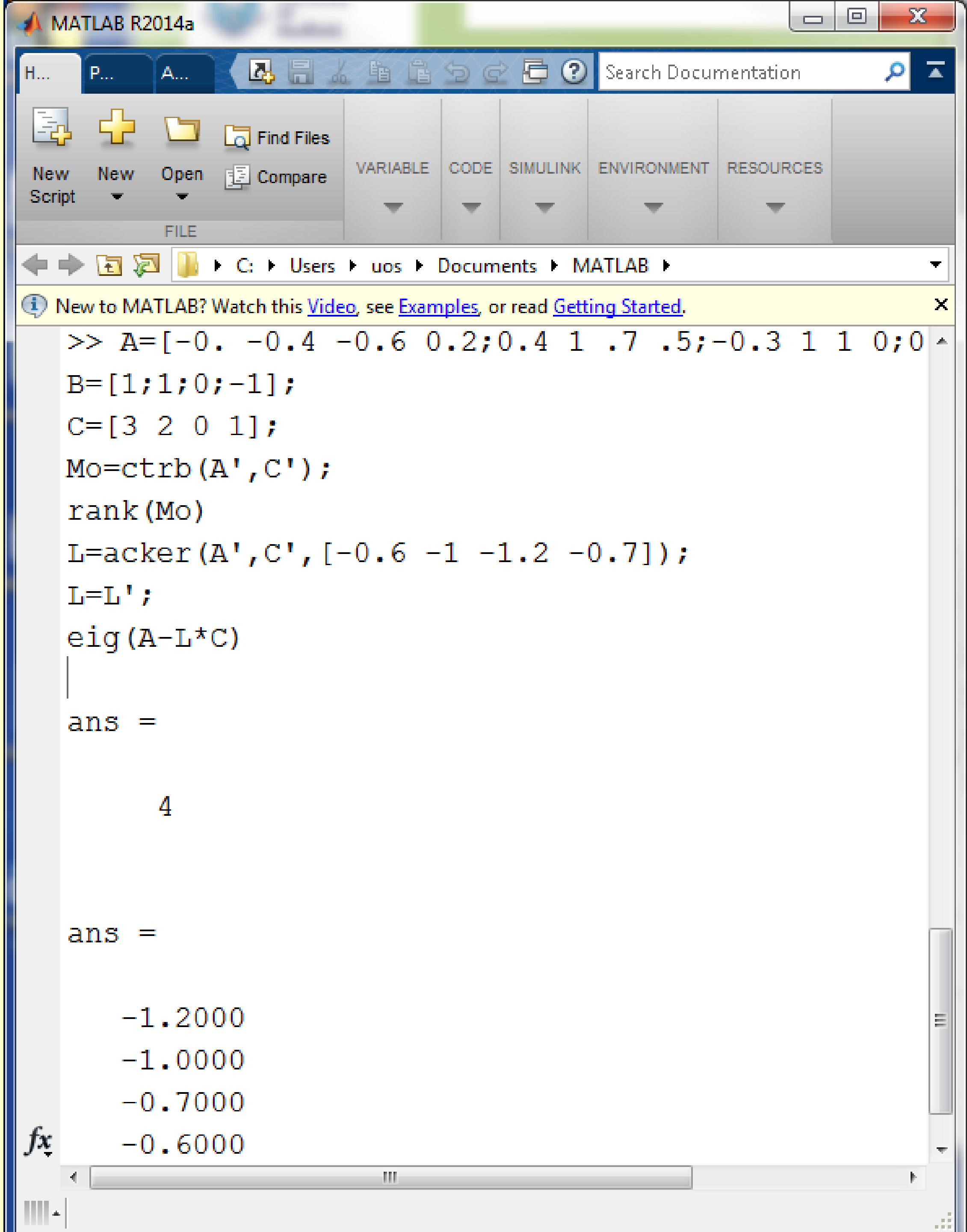
Ackermann's formulae for observer design



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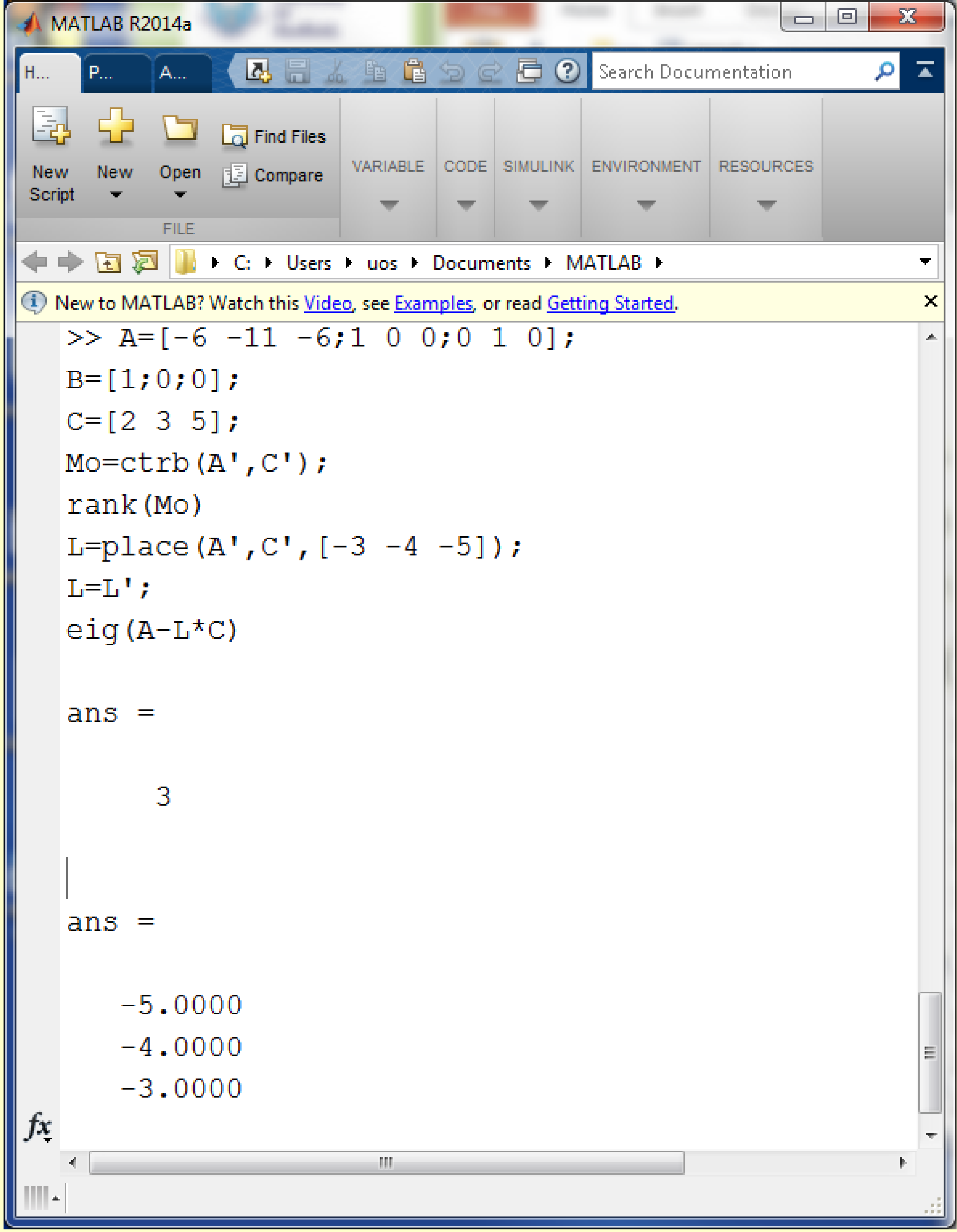
Using place.m to place poles of an observer



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Conclusion

Shown that one can use the same pole placement techniques used with state feedback to place the poles of an observer.

1. Shown the duality between the dynamics of:

$$\dot{x} = (A - BK)x \qquad \dot{e} = (A - LC)e$$

2. Use same pole placement algorithms but use A^T in place of A and C^T in place of B.
3. Require full observability in order to be able to select L, to place poles arbitrarily.

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Open Engineering Resources
HEA Engineering Subject Centre
A HEFCE/JISC/HEA Pilot Project

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